A generalization of Kruskal’s theorem

What happens when you replace the Kruskal ranks with standard ranks in Kruskal’s theorem?

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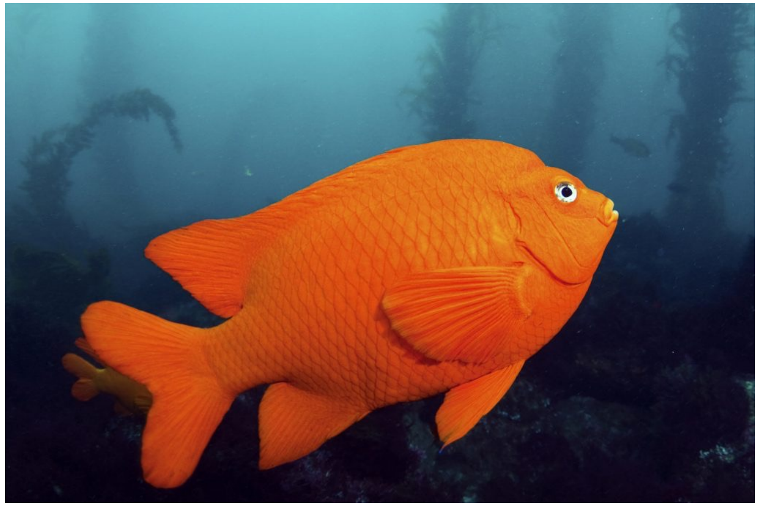
Outline

- Uniqueness of tensor decompositions
- Kruskal's theorem
- A generalization of Kruskal's theorem
- Splitting theorem



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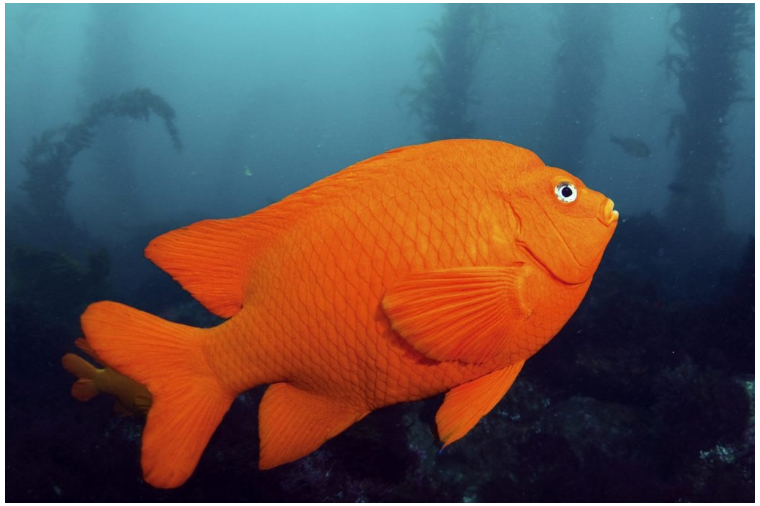

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What is a matrix?

<u>Definition:</u> Let $n \in \mathbb{N}$ and $[n] \coloneqq \{1, \dots, n\}$.

Let $\mathbb{F}$ be a field, and let X, Y, Z be $\mathbb{F}$-vector spaces of dimension at least 2.

A matrix is a 2-way array $\begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$

Think of $X \otimes Y$ as the set of $\dim(X) \times \dim(Y)$ matrices

Think of $x \otimes y$ as the array $xy^T = (x_i y_j)_{(i,j)}$

We say that $T \in X \otimes Y$ is product, or rank-one if $T = x \otimes y$ for some $x \in X, y \in Y$

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A ~~matrix~~ tensor is a 3-way array $\begin{bmatrix} 15 & 18 \\ 20 & 24 \end{bmatrix}$ $\begin{bmatrix} 30 & 36 \\ 40 & 48 \end{bmatrix}$

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We say that $T \in X \otimes Y \otimes Z$ is product, or rank-one if $T = x \otimes y \otimes z$ for some $x \in X, y \in Y, z \in Z$

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A ~~matrix~~ tensor is a 3-way array $\begin{bmatrix} 15 & 18 \\ 20 & 24 \end{bmatrix}$ $\begin{bmatrix} 30 & 36 \\ 40 & 48 \end{bmatrix}$ $= \binom{1}{2} \otimes \binom{3}{4} \otimes \binom{5}{6}$

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Let $\mathbb{F}$ be a field, and let X, Y, Z be $\mathbb{F}$-vector spaces of dimension at least 2.

For $T \in X \otimes Y \otimes Z$, an expression $$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z$$

is called a decomposition of T into product tensors

$$\operatorname{rank}(T) := \min\{ n : \text{there exists a decomposition of } T \text{ into } n \text{ product tensors}\}$$

Uniqueness of tensor decompositions

<u>Definition:</u> Let $n \in \mathbb{N}$ and $[n] := \{1, \ldots, n\}$.

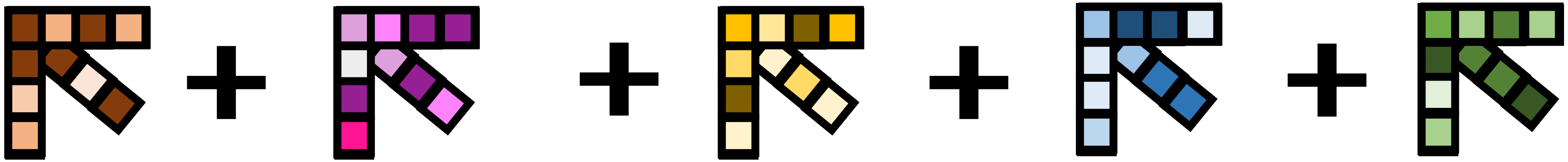
A tensor decomposition

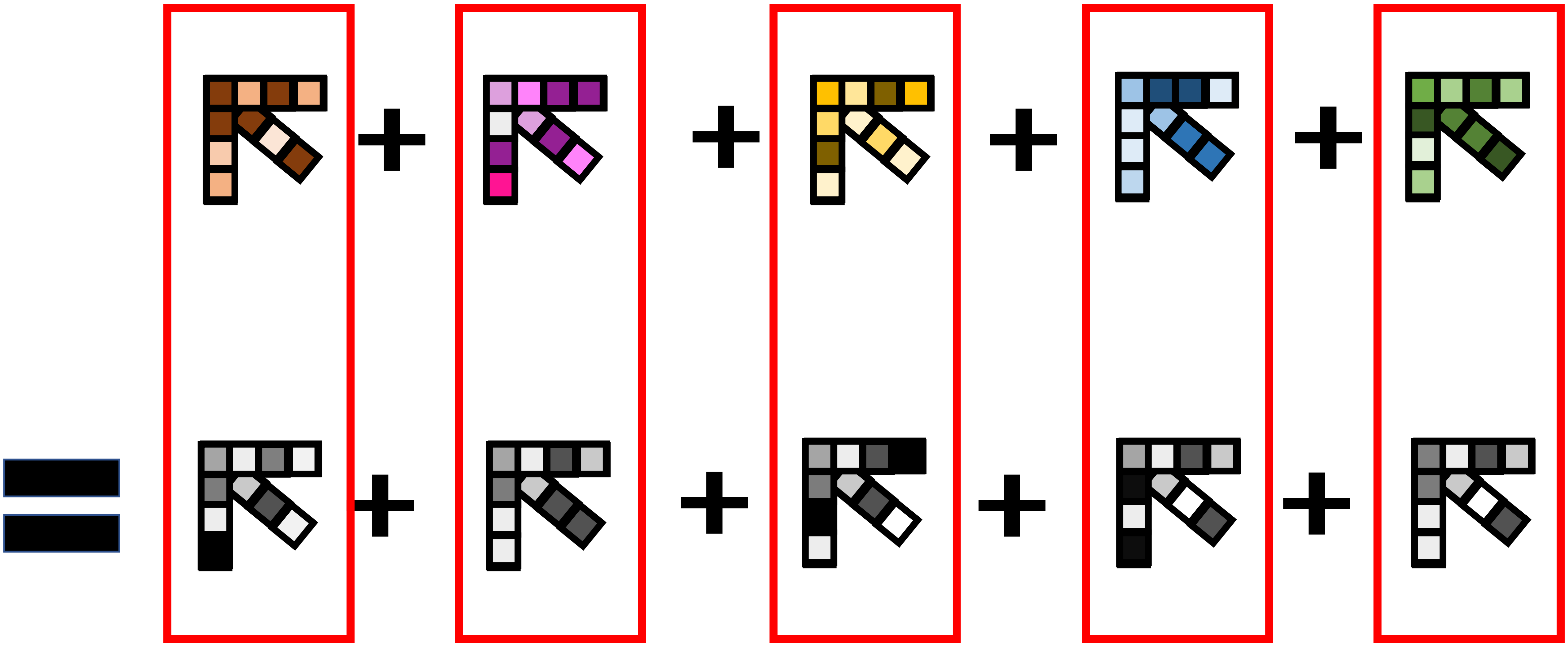
$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z$$

is called the unique (rank) decomposition of T if rank(T)=n and for any other decomposition

$$T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a \in X \otimes Y \otimes Z$$

there is a permutation $\sigma \in S_n$ such that $x_a \otimes y_a \otimes z_a = x'_{\sigma(a)} \otimes y'_{\sigma(a)} \otimes z'_{\sigma(a)}$ for all $a \in [n]$.





Applications

- Tensors ⟷ Physical data
- Tensor decomposition ⟷ Interpretation of data
- Unique decomposition ⟷ Unique interpretation

Example: Latent parameter learning

L is for *latent*

- Let A, B, C, L be discrete random variables such that A, B, C are conditionally independent, i.e.

$$\Pr(a, b, c|l) = \Pr(a|l)\Pr(b|l)\Pr(c|l) \quad \text{for all } a, b, c, l.$$

- Goal: Given the probability vector $\Pr(A, B, C)$, determine $\Pr(A, B, C, L)$.
- Method:

$$\Pr(A, B, C) = \sum_l \Pr(l)\Pr(A, B, C|l) = \sum_l \underbrace{\Pr(l)\Pr(A|l) \otimes \Pr(B|l) \otimes \Pr(C|l)}$$

… If $\Pr(A, B, C)$ has a unique decomposition, then we can recover $\Pr(A, B, C, l)$,

- Applications: Learning mixtures of spherical gaussians, phylogenetic tree reconstruction, hidden Markov models, orbit retrieval, blind signal separation, document topic models, …

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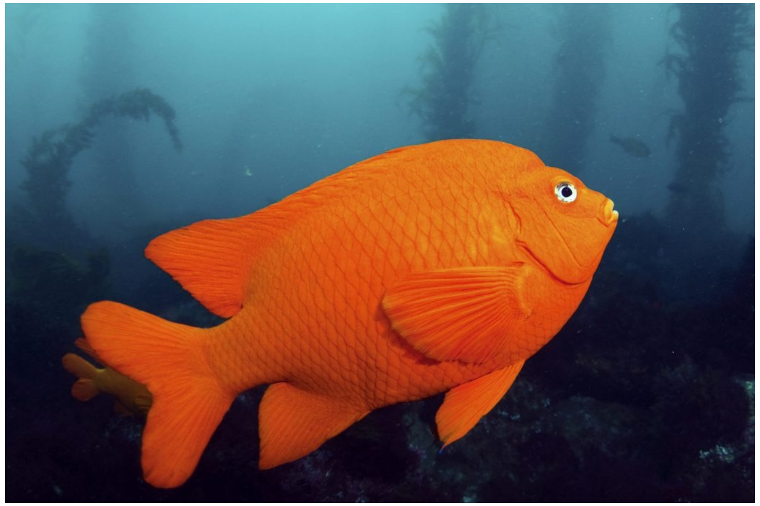

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Kruskal's theorem

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

General flavor of these results:

We are handed a decomposition (1), and we want to know if it is the unique decomposition of T.

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Notation: When $S \subseteq [n]$, $\quad d_x^S := \operatorname{dimspan}\{x_a : a \in S\}$

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$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} + e_4^{\otimes 3} + (e_1 + e_2) \otimes (e_2 + e_3) \otimes (e_1 + e_4)$$

$x_1 \otimes y_1 \otimes z_1$ … x_5 $\otimes$ y_5 $\otimes$ z_5

For $S = \{1,2,5\}$, $\quad d_x^S = 2,\ d_y^S = 3,\ d_z^S = 3$

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Definition: The Kruskal rank of $\{x_1, \ldots, x_n\} \in X$ is the largest integer k_x such that for every subset $S \subseteq [n]$ of size $|S| = k_x$, it holds that

$d_x^S = |S|$.

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$$\{x_1, \ldots, x_5\} = \{e_1, e_2, e_3, e_4, e_1 + e_2\}, \qquad k_x = 2,\ d_x^{[5]} = 4.$$

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Kruskal’s theorem: If $2n \leq k_x + k_y + k_z - 2$, then (1) is the unique decomposition of T.

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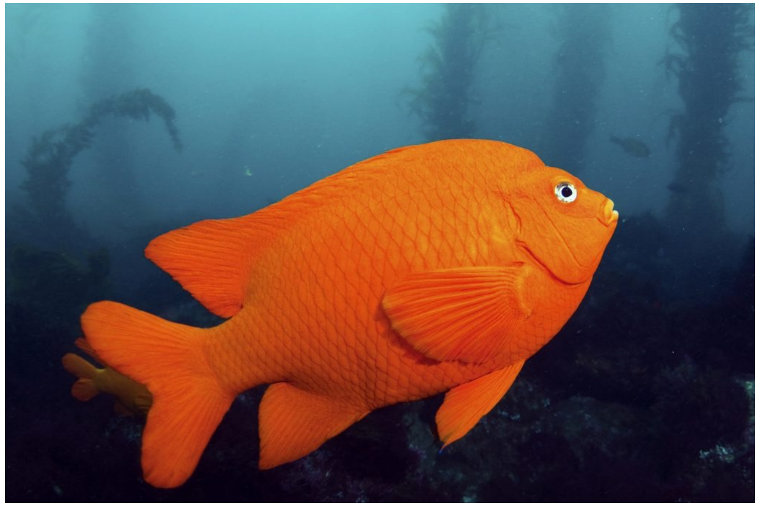

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Our generalization

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

Idea: Replace Kruskal ranks k_x with standard dimspans $d_x^{[n]}$.

Theorem [Gubkin-L-Petrov]: If

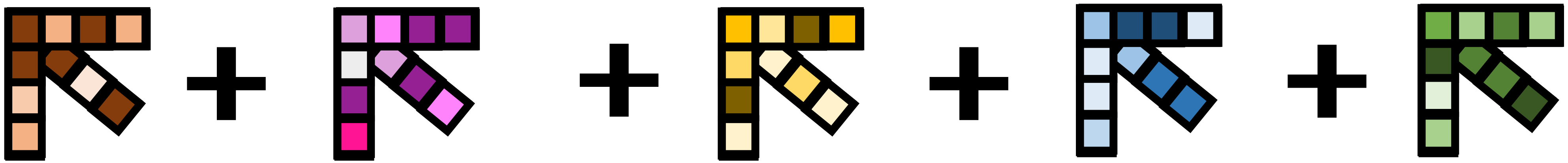
$d_x^{[n]} = \text{dimspan}\{x_1, \ldots, x_n\}$

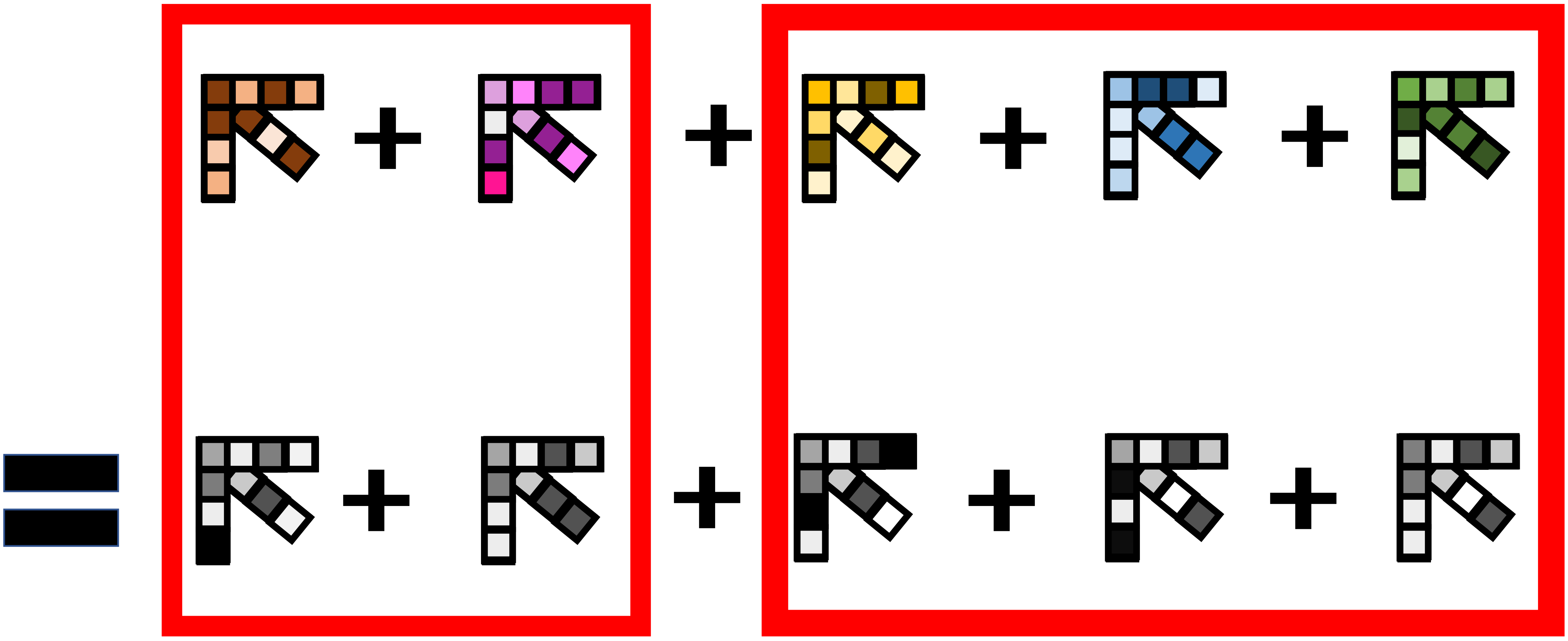
$$2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then for any other decomposition $T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a \in X \otimes Y \otimes Z$

there exist non-trivial subsets $S, R \subseteq [n]$ such that

$$\sum_{a \in S} x_a \otimes y_a \otimes z_a = \sum_{a \in R} x'_a \otimes y'_a \otimes z'_a$$





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<u>Theorem [Gubkin-L-Petrov]:</u> If for every subset $S \subseteq [n]$ of size $|S| \geq 2$,

it holds that

$$2|S| \leq d_x^S + d_y^S + d_z^S - 2,$$

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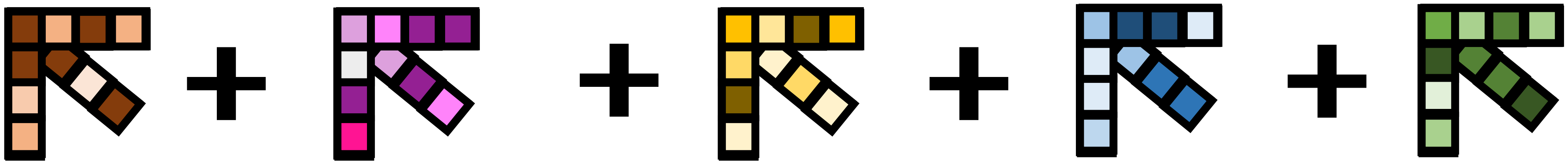

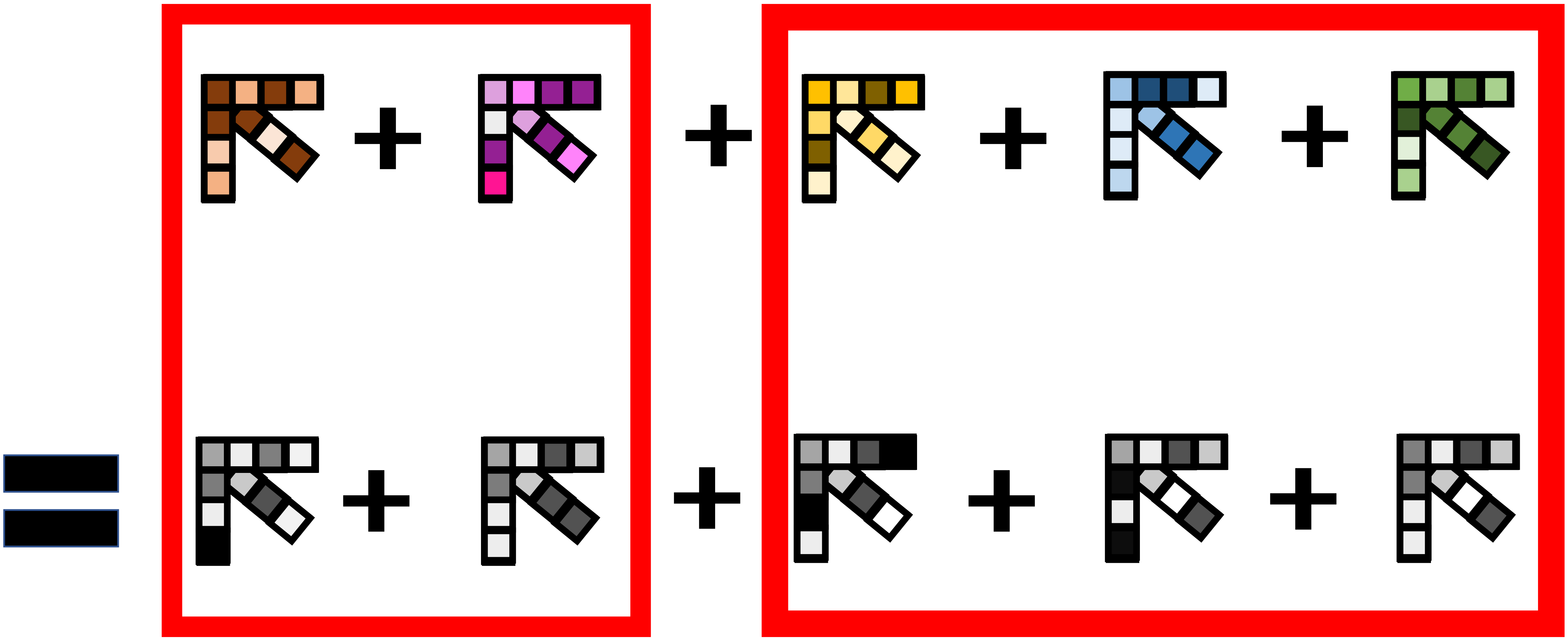

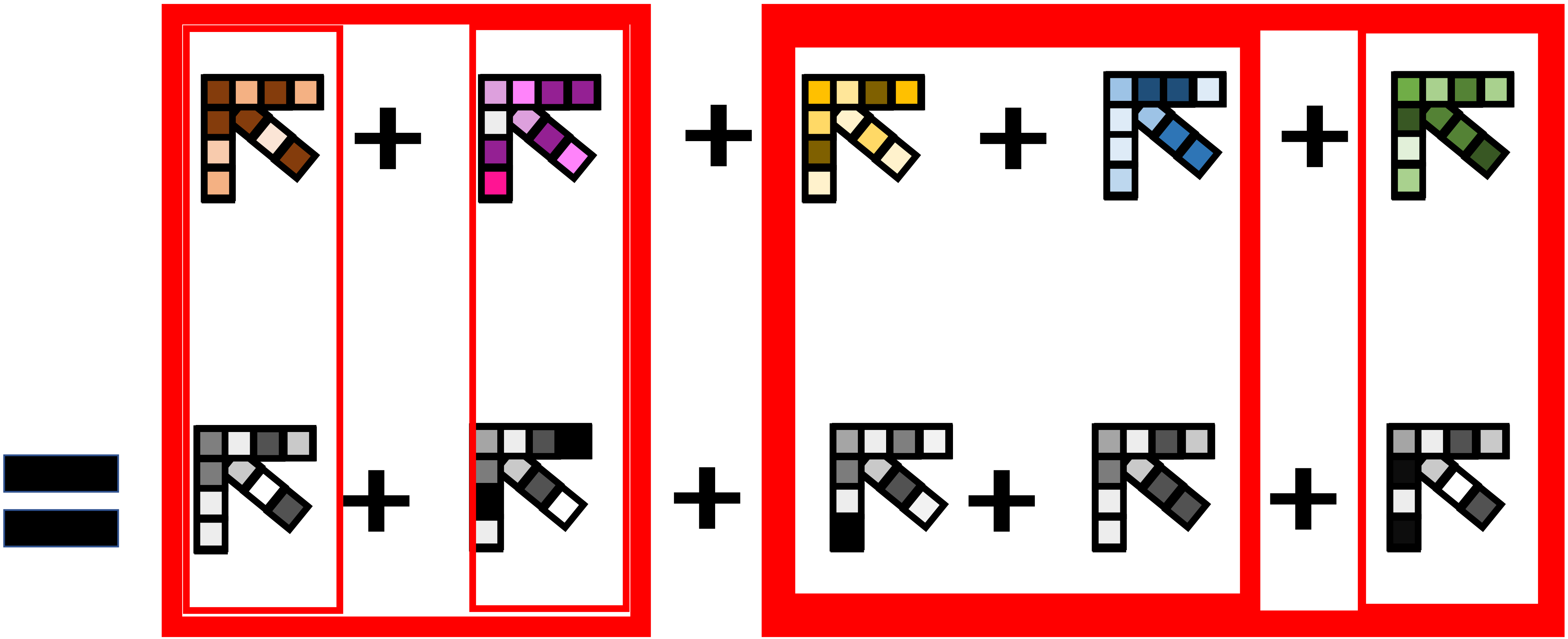

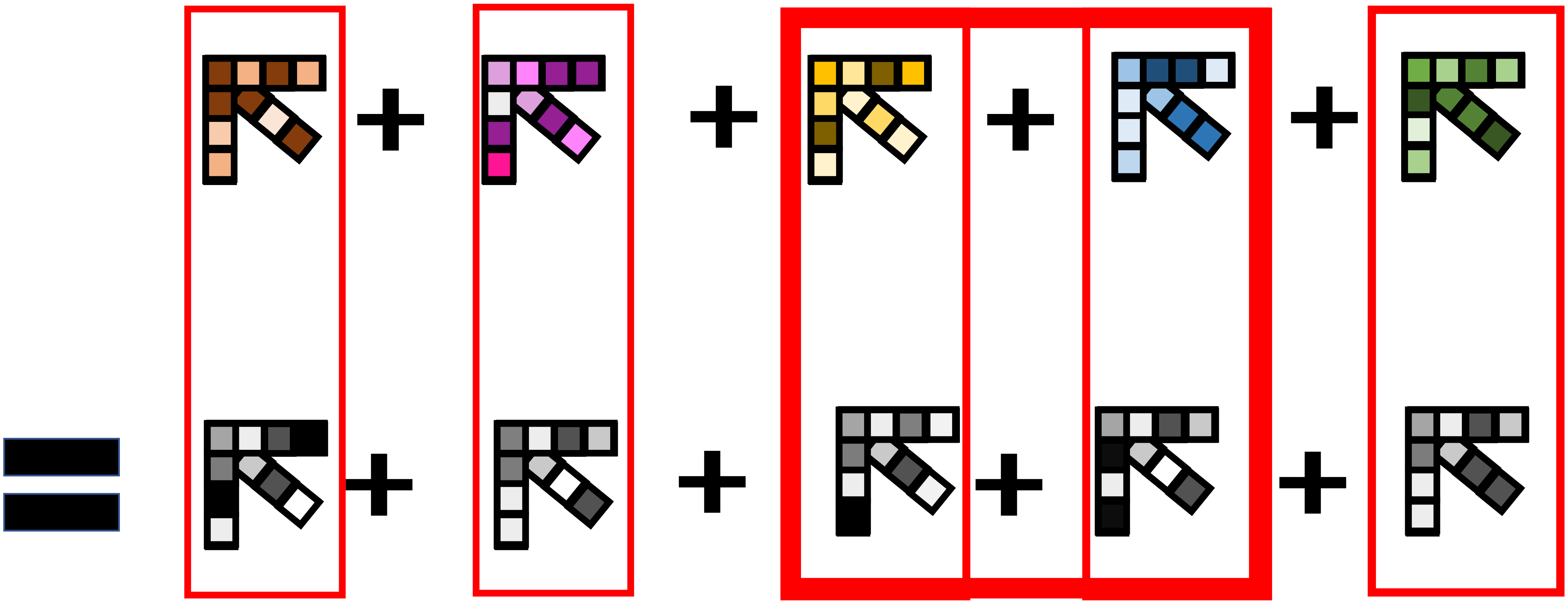

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Example

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} + e_4^{\otimes 3} + (e_1 + e_2) \otimes (e_2 + e_3) \otimes (e_1 + e_4)$$

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Kruskal's theorem does not certify uniqueness

$$10 = 2n \nleq k_x + k_y + k_z - 2 = 2 + 2 + 2 - 2 = 4$$

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For $S = \{1,2\}$, $4 = 2|S| \leq d_x^S + d_y^S + d_z^S - 2 = 2 + 2 + 2 - 2 = 4$

For $S = \{1,2,5\}$, $6 = 2|S| \leq d_x^S + d_y^S + d_z^S - 2 = 2 + 3 + 3 - 2 = 6$

For $S = \{1,2,3,5\}$, $8 = 2|S| \leq d_x^S + d_y^S + d_z^S - 2 = 3 + 3 + 4 - 2 = 8$

For $S = [5]$, $10 = 2|[n]| \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2 = 4 + 4 + 4 - 2 = 10$

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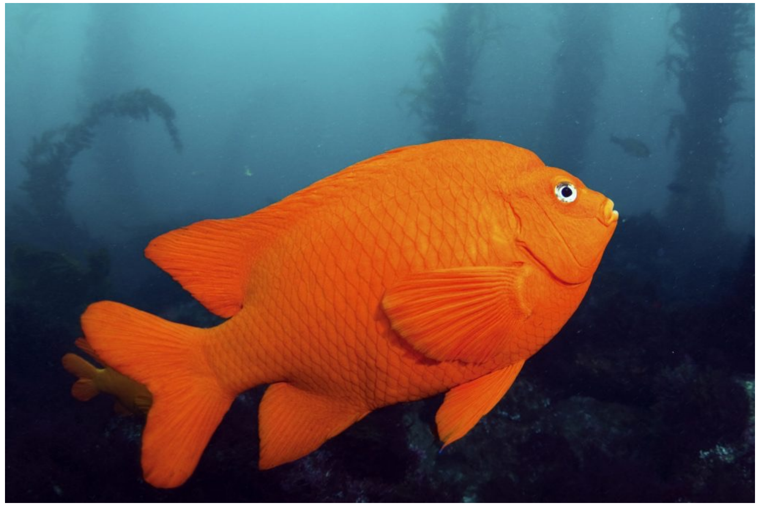

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Splitting

Definition: A set of vectors $E = \{v_1, \ldots, v_n\}$ splits if there exists a non-trivial subset $S \subseteq E$ such that

$$\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\} \qquad (2)$$

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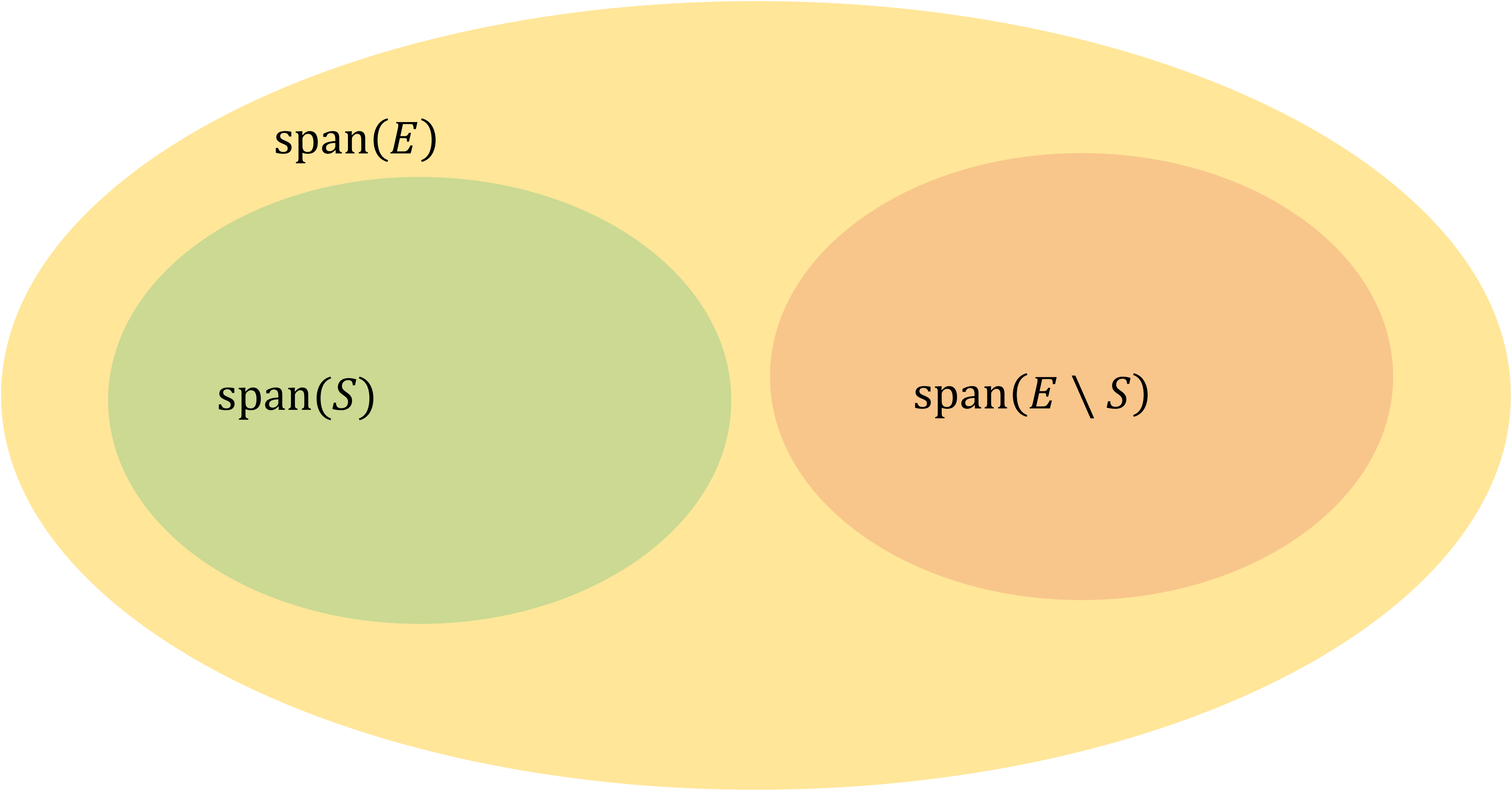

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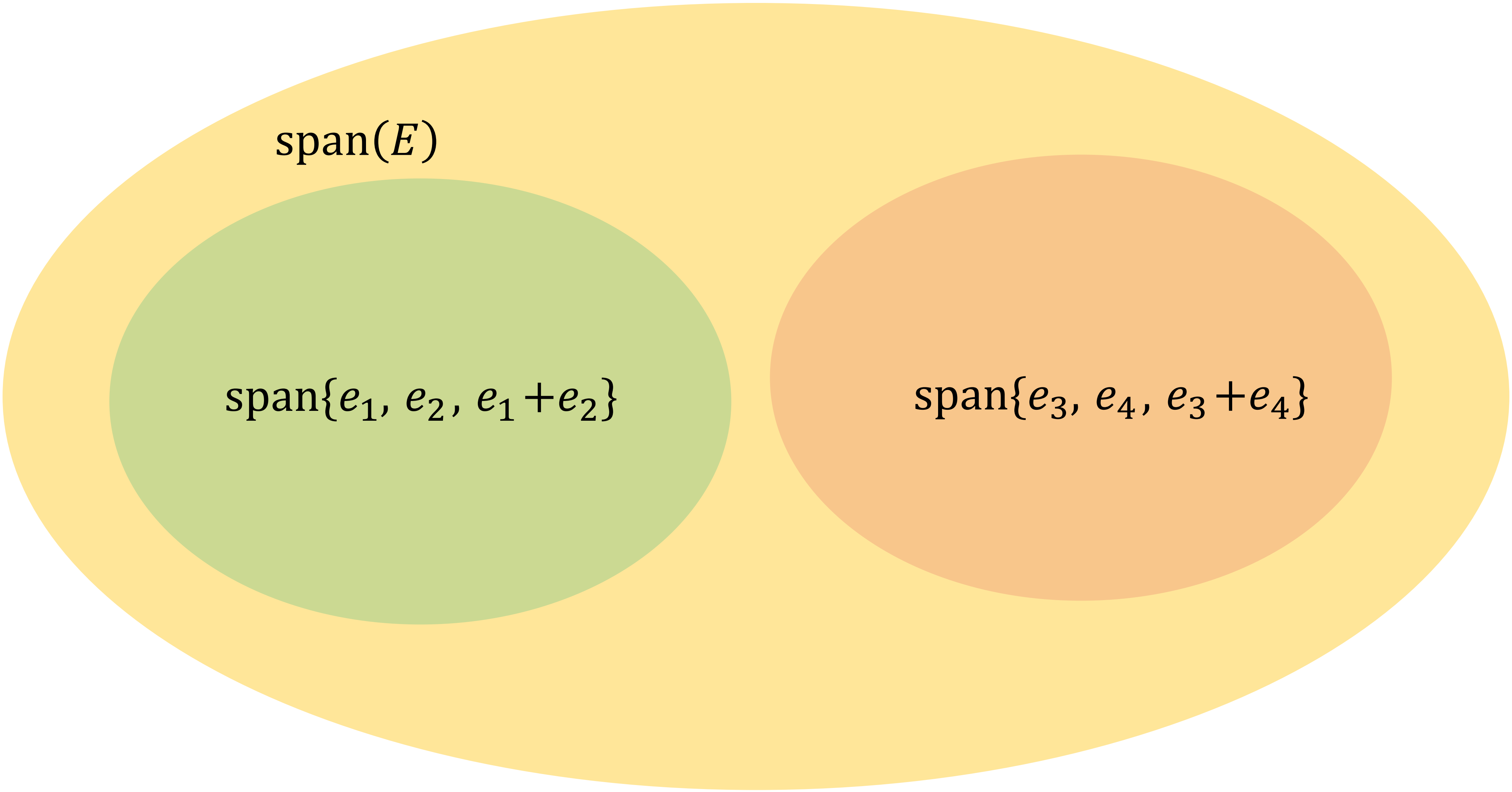
$E = \{e_1, e_2, e_1 + e_2, e_3, e_4, e_3 + e_4\}$



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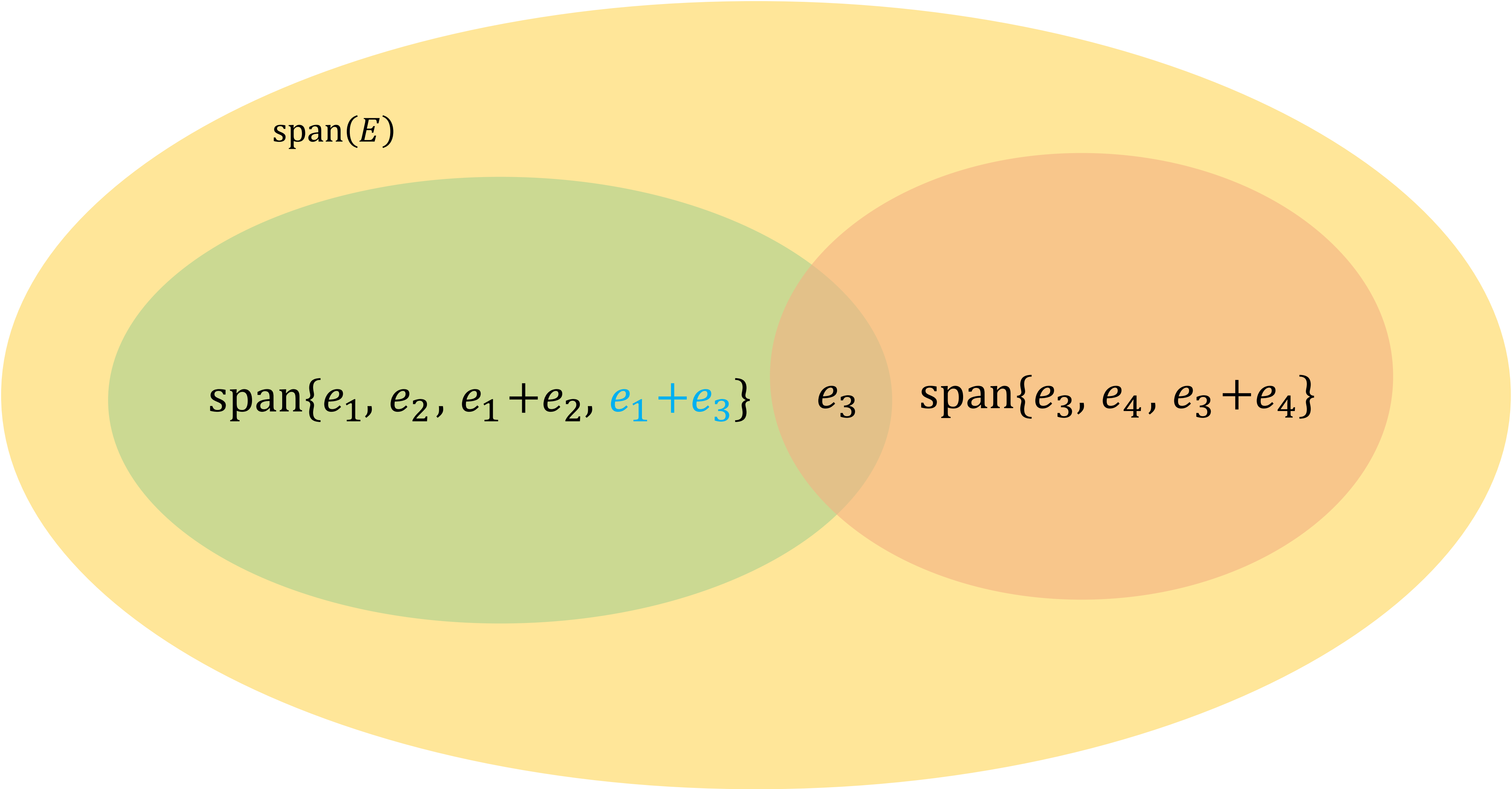
$E = \{e_1, e_2, e_1+e_2, e_3, e_4, e_3+e_4\} \cup \{e_1+e_3\}$



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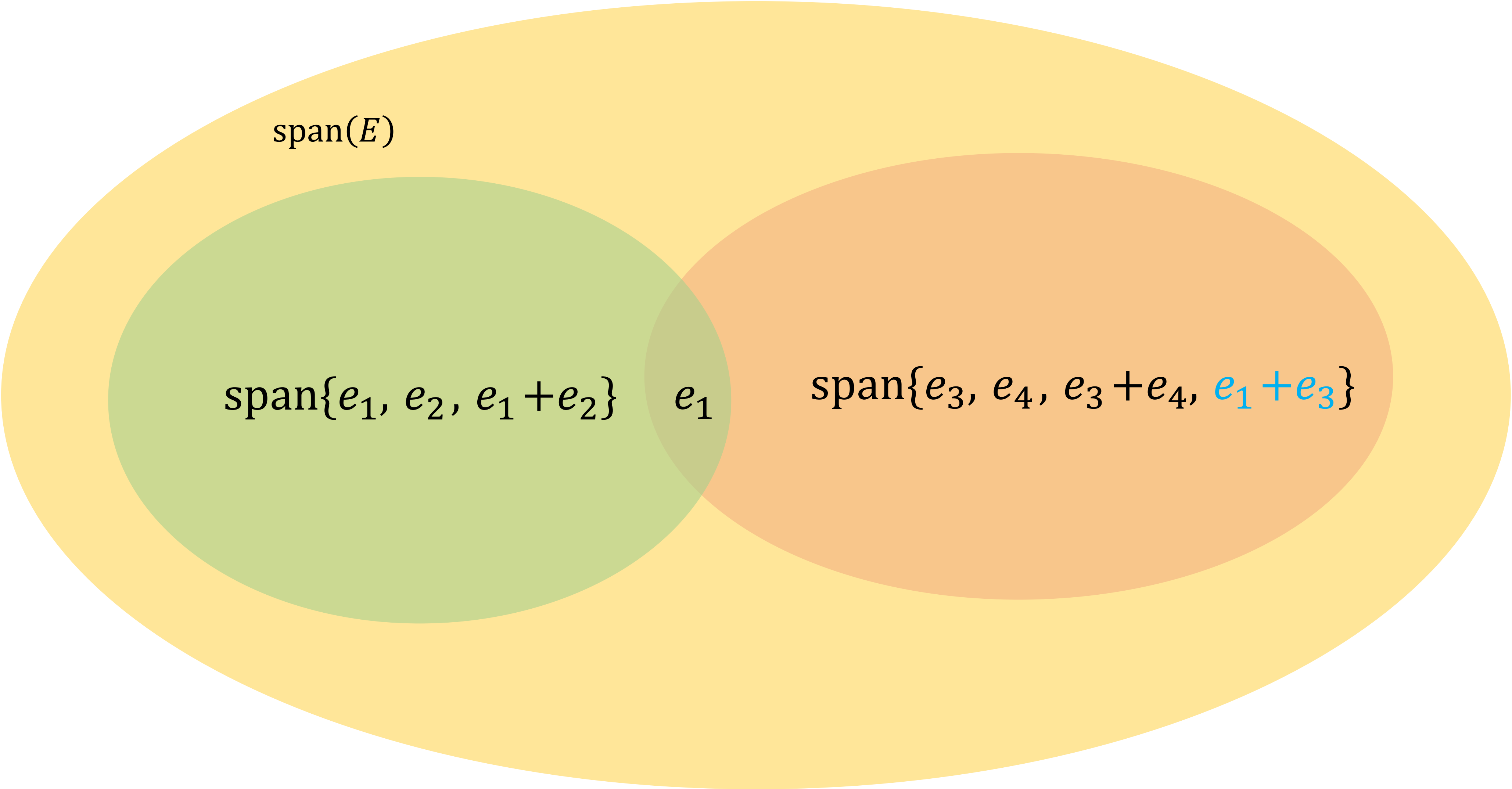

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Fact: If (2) holds, and $\sum(E) = 0$, then $\sum(S) = \sum(E \setminus S) = 0$

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Fact: If (2) holds, and $\sum(E) = 0$, then $\sum(S) = \sum(E \setminus S) = 0$

Proof: $\sum(E) = 0 \Rightarrow \sum(S) = -\sum(E \setminus S) \in \mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

Splitting theorem

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Splitting theorem [Gubkin-L-Petrov]: Let $E = \{x_a \otimes y_a : a \in [n]\}$.

If

$$\mathrm{dimspan}(E) \leq d_x^{[n]} + d_y^{[n]} - 2$$

then E splits.

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(Our generalization of Kruskal's theorem is a corollary to this)

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then E splits.

Corollary: If $2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2$, then for any other set of product tensors $E' = \{x'_a \otimes y'_a \otimes z'_a : a \in [n]\}$, $E \cup E'$ splits.

Corollary => Kruskal generalization

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

Suffices to prove:

Theorem [Gubkin-L-Petrov]: If

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then for any other decomposition $T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a \in X \otimes Y \otimes Z$

there exist non-trivial subsets $S, R \subseteq [n]$ such that $\sum_{a \in S} x_a \otimes y_a \otimes z_a = \sum_{a \in R} x'_a \otimes y'_a \otimes z'_a$

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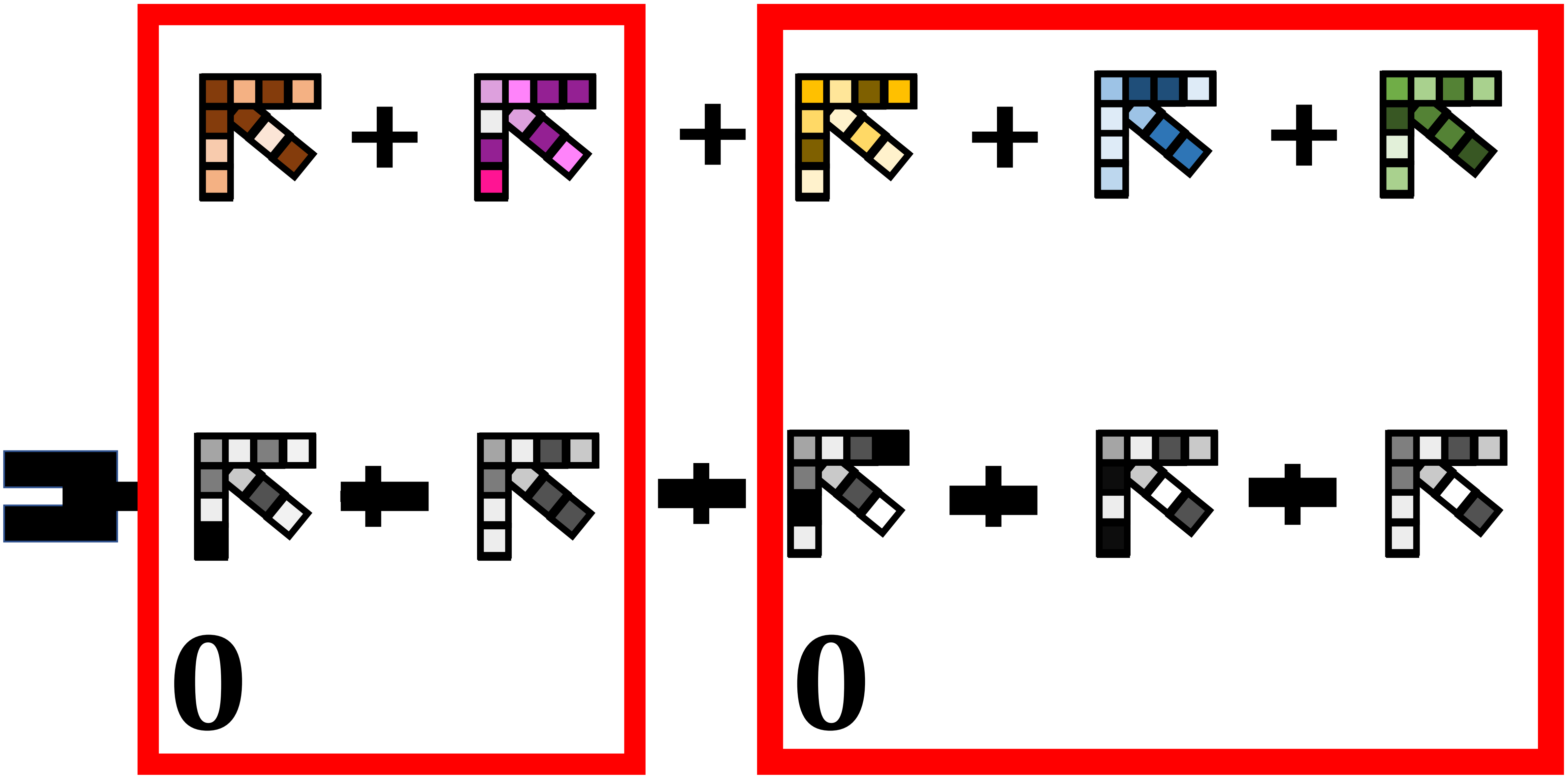
Proof:

By previous corollary, $\{x_a \otimes y_a \otimes z_a, x'_a \otimes y'_a \otimes z'_a \;: a \in [n]\}$ splits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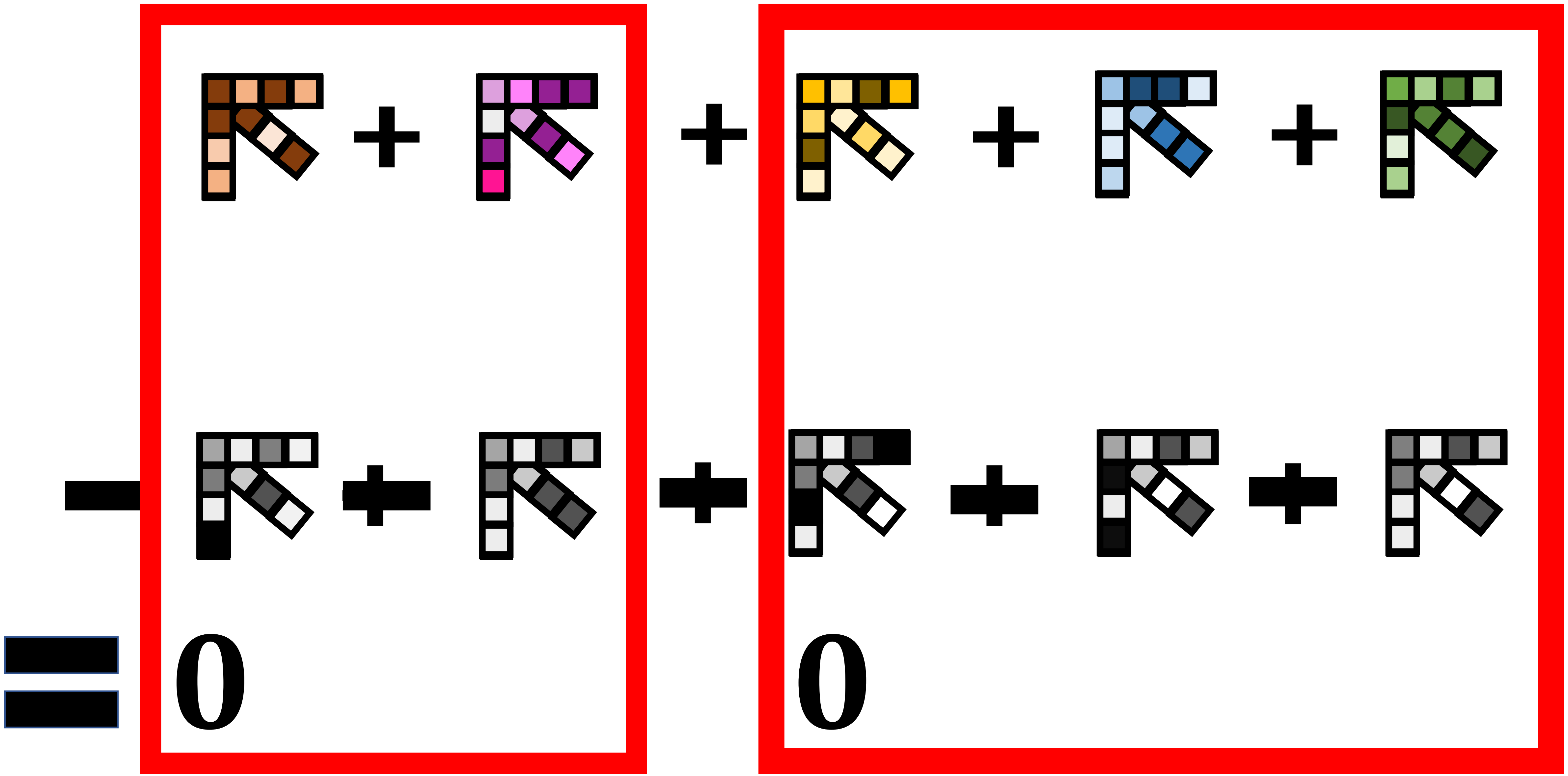
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Conclusion

- Uniqueness of tensor decompositions
- Kruskal’s theorem
- A generalization of Kruskal’s theorem
- Splitting theorem
- More matroid theory for product tensors?

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