A complete hierarchy of linear systems for certifying quantum entanglement of subspaces

Nathaniel Johnston[1] Benjamin Lovitz[2] Aravindan Vijayaraghavan[3]

1. Mount Allison University
2. NSF Postdoc, Northeastern University
3. Northwestern University

QIP 2023

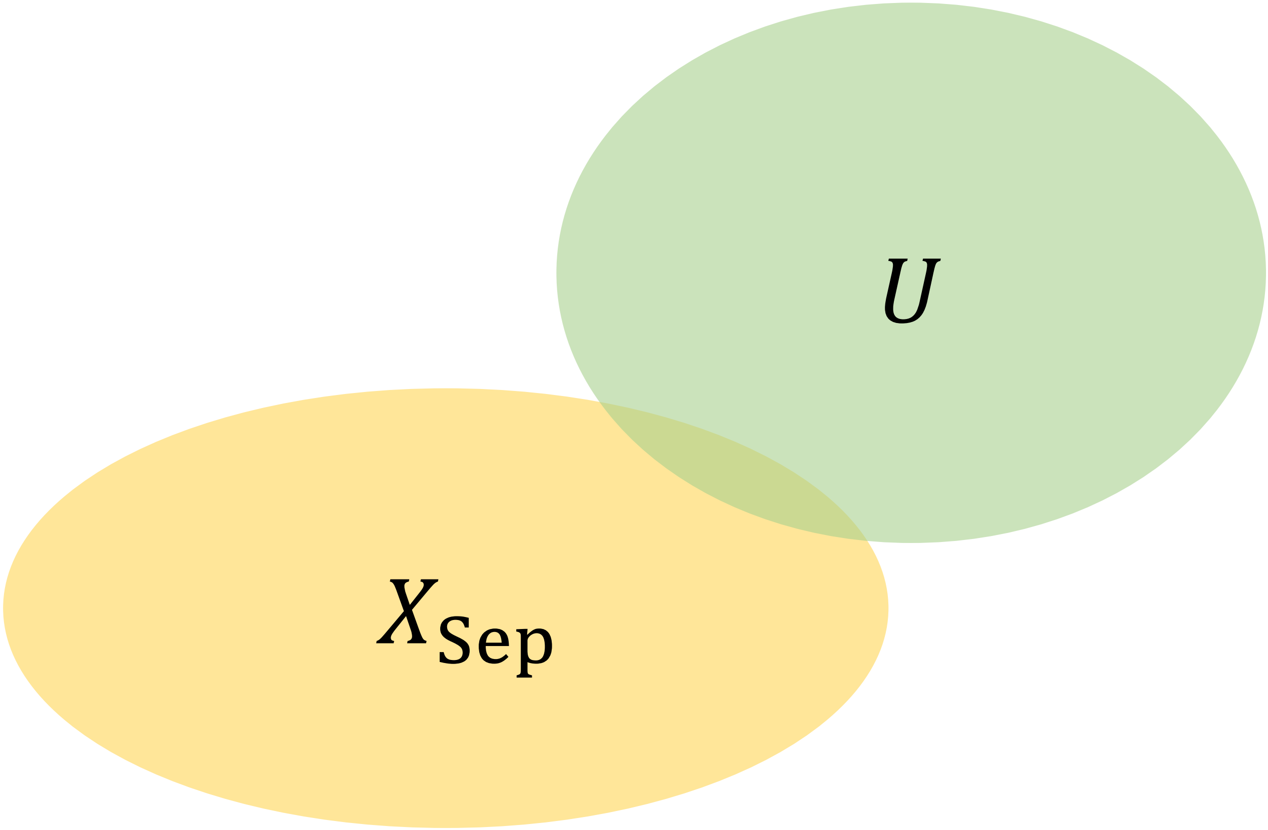
February 6, 2023

Northeastern University

NSF

Product pure states: $X_{\text{Sep}} = \{|\psi\rangle \otimes |\phi\rangle : |\psi\rangle, |\phi\rangle \in \mathbb{C}^d\} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$, determine if U is entangled, i.e. if $U \cap X_{\text{Sep}} = \{0\}$.

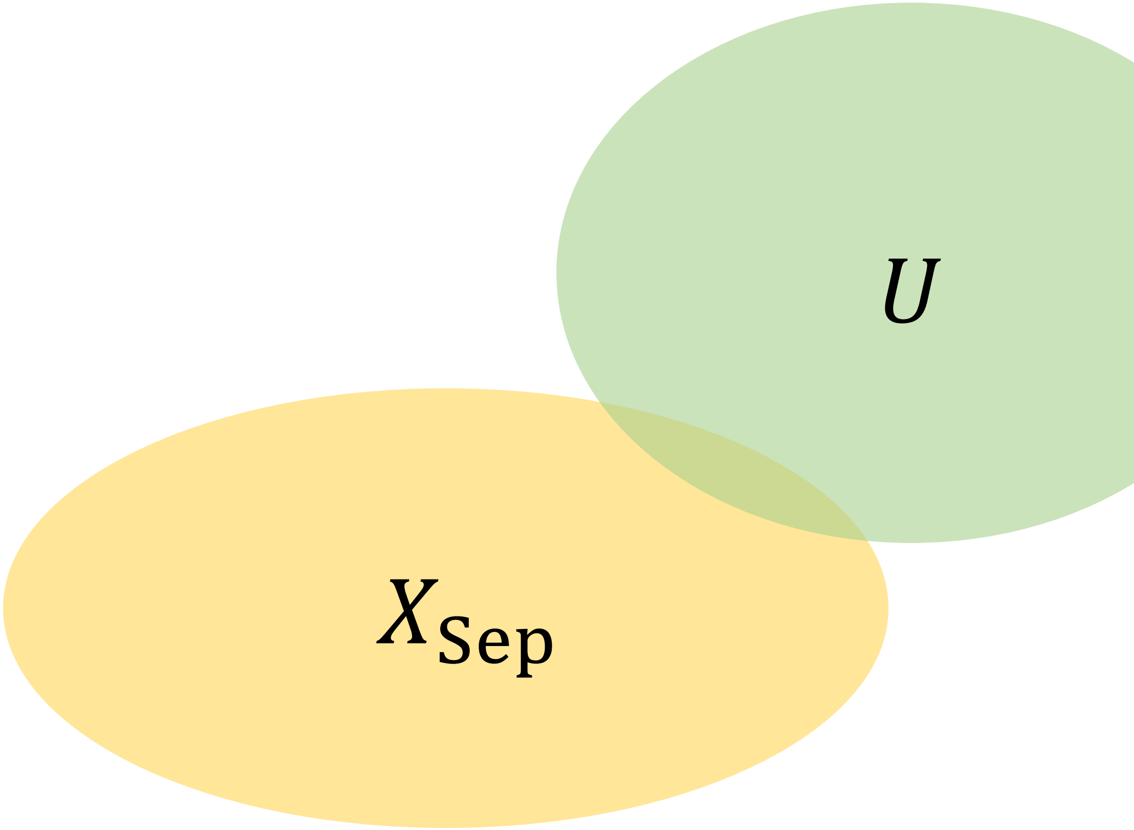


Applications:

- Range criterion: For a density operator $\rho \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$,
  $\mathrm{Im}(\rho)$ entangled $\Rightarrow$ ρ entangled
- Entangled subspaces can be used to construct entanglement witnesses and quantum error-correcting codes

Product pure states: $X_{\mathrm{Sep}} = \{|\psi\rangle \otimes |\phi\rangle : |\psi\rangle, |\phi\rangle \in \mathbb{C}^d\} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$, determine if U is entangled, i.e. if $U \cap X_{\mathrm{Sep}} = \{0\}$.



Outline:

- Algorithm
- Complete hierarchy
- Extension to other notions of entanglement
- Robust generalization of hierarchy
- Extending symmetric extensions [DPS 04] to other notions of separability

Product pure states: $X_{\text{Sep}} = \{|\psi\rangle \otimes |\phi\rangle : |\psi\rangle, |\phi\rangle \in \mathbb{C}^d\} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$,
determine if U is entangled, i.e. if $U \cap X_{\text{Sep}} = \{0\}$.

[Buss et al 1999]: This is NP-Hard in the worst case.

[Barak et al 2019]: Best known algorithm takes $2^{\tilde{O}(\sqrt{d})}$ time.

[Classical AG, Parthasarathy 01]: $\dim(U) > (d-1)^2 \Rightarrow$ U is not entangled

U generic and $\dim(U) \leq (d-1)^2 \Rightarrow$ U is entangled

Algorithm [JLV 22]: Takes poly(d)-time and outputs either: "Hay in a haystack problem"

1. Fail, or
2. A certificate that U is entangled

Works-Extremely-Well Theorem [JLV 22]:

U generic and $\dim(U) \leq \frac{1}{4}(d-1)^2 \Rightarrow$ Algorithm outputs a certificate that U is entangled

Product pure states: $X_{\text{Sep}} = \{|\psi\rangle \otimes |\phi\rangle : |\psi\rangle, |\phi\rangle \in \mathbb{C}^d\} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$,
determine if U is entangled, i.e. if $U \cap X_{\text{Sep}} = \{0\}$.

[Buss et al 1999]: This is NP-Hard in the worst case.

[Barak et al 2019]: Best known algorithm takes $2^{\tilde{O}(\sqrt{d})}$ time.

[Classical AG, Parthasarathy 01]: $\dim(U) > (d-1)^2 \Rightarrow$ U is not entangled

U generic and $\dim(U) \leq (d-1)^2 \Rightarrow$ U is entangled

Algorithm [JLV 22]: Takes poly(d)-time and outputs either: "Hay in a haystack problem"

1. Fail, or
2. A certificate that U is entangled, or
3. **A separable state $|\boldsymbol{\psi}\rangle \otimes |\boldsymbol{\phi}\rangle \in \boldsymbol{U}$.**

[JLV 22]: WEW Theorem when U contains a generic planted separable state

Works-Extremely-Well Theorem [JLV 22]:

U generic and $\dim(U) \leq \frac{1}{4}(d-1)^2 \Rightarrow$ Algorithm outputs a certificate that U is entangled

Product pure states: $X_{\text{Sep}} = \{|\psi\rangle \otimes |\phi\rangle : |\psi\rangle, |\phi\rangle \in \mathbb{C}^d\} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$, determine if U is entangled, i.e. if $U \cap X_{\text{Sep}} = \{0\}$.

[Buss et al 1999]: This is NP-Hard in the worst case.

[Barak et al 2019]: Best known algorithm takes $2^{\tilde{O}(\sqrt{d})}$ time.

[Classical AG, Parthasarathy 01]: $\dim(U) > (d-1)^2 \Rightarrow U$ is not entangled

U generic and $\dim(U) \le (d-1)^2 \Rightarrow U$ is entangled

Algorithm [JLV 22]: Takes poly(d)-time and outputs either:

"Hay in a haystack problem"

1. Fail, or
2. A certificate that U is entangled, or
3. ~~A separable state $|\psi\rangle \otimes |\phi\rangle \in U$.~~

This talk: Focus on certification part of algorithm

~~[JLV 22]: WEW Theorem when U contains a generic planted separable state~~

Works-Extremely-Well Theorem [JLV 22]:

U generic and $\dim(U) \le \frac{1}{4}(d-1)^2 \Rightarrow$ Algorithm outputs a certificate that U is entangled

The Algorithm

Product pure states: $X_{\mathrm{Sep}} = \{|\psi\rangle \otimes |\phi\rangle : |\psi\rangle, |\phi\rangle \in \mathbb{C}^d\} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$, determine if U is entangled, i.e. if $U \cap X_{\mathrm{Sep}} = \{0\}$.

Idea: Problem is difficult because it's non-linear ($X_{\mathrm{Sep}} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$ isn't a linear subspace).

Make it linear: Instead check if $U \cap \mathrm{Span}(X_{\mathrm{Sep}}) = \{0\}$.

Doesn't work: $\mathrm{Span}(X_{\mathrm{Sep}}) = \mathbb{C}^d \otimes \mathbb{C}^d$.

Lift it up: Let $X^2_{\mathrm{Sep}} = \mathrm{Span}\{(|\psi\rangle \otimes |\phi\rangle)^{\otimes 2} : |\psi\rangle, |\phi\rangle \in \mathbb{C}^d\} = S^2(\mathbb{C}^d) \otimes S^2(\mathbb{C}^d)$

Check if $U^{\otimes 2} \cap X^2_{\mathrm{Sep}} = \{0\}$. Works extremely well!

Product pure states: $X_{\text{Sep}} = \{|\psi\rangle \otimes |\phi\rangle : |\psi\rangle, |\phi\rangle \in \mathbb{C}^d\} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$,
determine if U is entangled, i.e. if $U \cap X_{\text{Sep}} = \{0\}$.

Let $X_{\text{Sep}}^2 = \text{Span}\{(|\psi\rangle \otimes |\phi\rangle)^{\otimes 2} : |\psi\rangle, |\phi\rangle \in \mathbb{C}^d\} = S^2(\mathbb{C}^d) \otimes S^2(\mathbb{C}^d)$

Takes poly(d) time to check

Algorithm:
If $U^{\otimes 2} \cap X_{\text{Sep}}^2 = \{0\}$, output U is entangled
Otherwise, output Fail

Correctness: $|\psi\rangle \otimes |\phi\rangle \in U \Rightarrow (|\psi\rangle \otimes |\phi\rangle)^{\otimes 2} \in U^{\otimes 2} \cap X_{\text{Sep}}^2$
$\Rightarrow$ Algorithm outputs Fail.

Algorithm runtime to certify $U \cap X_{\mathbf{Sep}} = \{0\}$

d	$\dim(U)$	time
3	3	0.01 s
4	8	0.03 s
5	13	0.08 s
6	20	0.20 s
7	29	0.49 s
8	39	1.06 s
9	50	2.24 s
10	63	5.56 s

From algorithm
to complete hierarchy
(based on Hilbert’s Nullstellensatz)

Product pure states: $X_{\text{Sep}} = \{|\psi\rangle \otimes |\phi\rangle : |\psi\rangle, |\phi\rangle \in \mathbb{C}^d\} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$,
determine if U is entangled, i.e. if $U \cap X_{\text{Sep}} = \{0\}$.

Let $X_{\text{Sep}}^k = \text{Span}\{(|\psi\rangle \otimes |\phi\rangle)^{\otimes k} : |\psi\rangle, |\phi\rangle \in \mathbb{C}^d\} = S^k(\mathbb{C}^d) \otimes S^k(\mathbb{C}^d)$

Algorithm k:

If $U^{\otimes k} \cap X_{\text{Sep}}^k = \{0\}$, output U is entangled

Otherwise, output Fail

Completeness [Hilbert]: For $k = 3^{d^2}$, Fail $\Leftrightarrow$ U not entangled

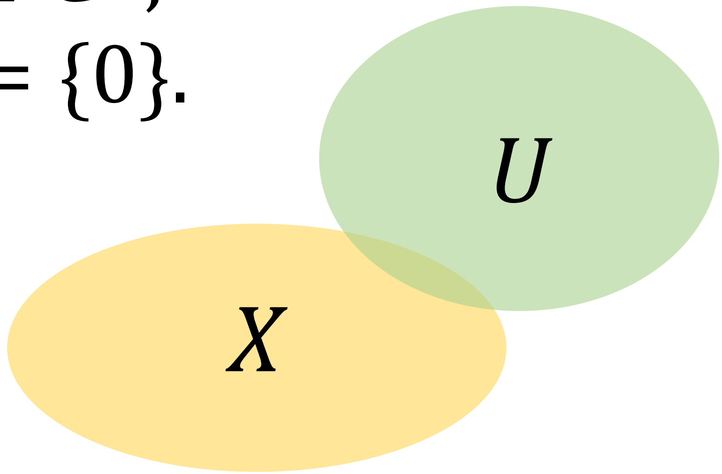
Analogous hierarchies for other notions of entanglement

Let $X \subseteq \mathbb{C}^D$ be nice* (for example, $X = X_{\text{Sep}} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$)

Think: "Some set of low entanglement"

*Any conic variety

<u>Problem:</u> Given a basis for a linear subspace $U \subseteq \mathbb{C}^D$,
determine if U avoids X, i.e. if $U \cap X = \{0\}$.

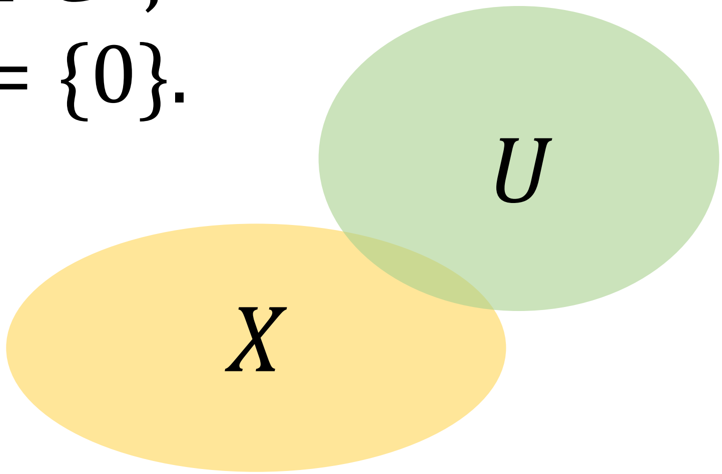


Let $X \subseteq \mathbb{C}^D$ be nice* (for example, $X = X_{\text{Sep}} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$)

Think: “Some set of low entanglement”

*Any conic variety

<u>Problem:</u> Given a basis for a linear subspace $U \subseteq \mathbb{C}^D$,
determine if U avoids X, i.e. if $U \cap X = \{0\}$.



Let $X^k = \text{Span}\{|\psi\rangle^{\otimes k} : |\psi\rangle \in X\}$

<u>Algorithm k:</u>

Takes $D^{O(k)}$ time to check

If $U^{\otimes k} \cap X^k = \{0\}$, output U avoids X

Otherwise, output Fail

<u>Completeness [Hilbert]:</u> For $k = 2^{O(D)}$, Fail $\Leftrightarrow$ U intersects X

Examples

WEW Theorem [JLV 22]: For generic U of dimension $\dim(U) \leq$
it holds that $U^{\otimes k} \cap X^k = \{0\}$ for $k =$

Schmidt rank $\leq r$ states

$X_r = \{|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d : \text{Schmidt–rank}(|\psi\rangle) \leq r\}$

$= \Omega_r(d^2)$
$= r + 1$

Product states

in-X_{Sep}-arable ↔ Completely entangled

$X_{\text{Sep}} = \{|\psi_1\rangle \otimes \cdots \otimes |\psi_m\rangle : |\psi_i\rangle \in \mathbb{C}^d\}$

$\sim (1/4) d^m$
$= 2$

Biseparable states

in-X_B-arable ↔ Genuinely entangled

$X_B = \{|\psi\rangle \in \left(\mathbb{C}^d\right)^{\otimes m}$: Some bipartition of $|\psi\rangle$ has rank 1$\}$

$\sim (1/4) d^m$
$= 2$

Slice rank 1 states

$X_S = \{|\psi\rangle \in (\mathbb{C}^n)^{\otimes m}$: Some 1 v.s. rest bipartition of $|\psi\rangle$ has rank 1$\}$

$\sim (1/4) d^m$
$= 2$

Matrix product states of bond dimension $\leq r$

$X_S = \{|\psi\rangle \in (\mathbb{C}^n)^{\otimes m}$: Every left-right bipartition has rank $\leq r\}$

$= \Omega_r(d^m)$
$= r + 1$

Examples

WEW Theorem [JLV 22]: For generic U of dimension $\dim(U) \leq$ 👽 it holds that $U^{\otimes k} \cap X^k = \{0\}$ for $k =$ 🐜

Schmidt rank $\leq r$ states

$X_r = \{|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d: \text{Schmidt–rank}(|\psi\rangle) \leq r\}$

👽 $= \Omega_r(d^2)$
🐜 $= r + 1$

Product states in-X_{Sep}-arable ↔ Completely entangled

$X_{\text{Sep}} = \{|\psi_1\rangle \otimes \cdots \otimes |\psi_m\rangle : |\psi_i\rangle \in \mathbb{C}^d\}$

👽 $\sim(1/4)d^m$
🐜 $= 2$

Biseparable states in-X_B-arable ↔ Genuinely entangled

$X_B = \{|\psi\rangle \in \left(\mathbb{C}^d\right)^{\otimes m}$: Some bipartition of $|\psi\rangle$ has rank 1$\}$

👽 $\sim(1/4)d^m$
🐜 $= 2$

Slice rank

$X_S = \{|\psi\rangle$

Takeaway: Algorithm certifies entanglement of subspaces of dimension a constant multiple of the maximum possible in polynomial time.

Matrix product states of bond dimension $\leq r$

$X_S = \{|\psi\rangle \in (\mathbb{C}^n)^{\otimes m}$: Every left-right bipartition has rank $\leq r\}$

👽 $= \Omega_r(d^m)$
🐜 $= r + 1$

Robust generalization of the hierarchy

Robust generalization:

Instead of determining whether U avoids X,

Compute $h_X(U) := \max_{|\psi\rangle \in X} \langle\psi|P_U|\psi\rangle$

$P_U = \mathrm{Proj}(U)$

U avoids $X \iff h_X(U) < 1$

Theorem/Robust Hierarchy [JLV 23+]:

Let $X \subseteq \mathbb{C}^D$ be nice*, $U \subseteq \mathbb{C}^D$ linear, and $P_U = \mathrm{Proj}(U)$.

*Any conic variety

For each k, let $\mu_k = \lambda_{\max}\left(P_X^k\left(P_U \otimes I^{\otimes k-1}\right)P_X^k\right)$. ← $P_X^k = \mathrm{Proj}(X^k)$

Then the μ_k form a non-increasing sequence converging to $h_X(U) := \max_{|\psi\rangle \in X} \langle\psi|P_U|\psi\rangle$.

Robust generalization:

Instead of determining whether U avoids X,

Compute $h_X(U) := \max_{|\psi\rangle \in X} \langle\psi|P_U|\psi\rangle$

$P_U = \mathrm{Proj}(U)$

U avoids $X \iff h_X(U) < 1$

Theorem/Robust Hierarchy [JLV 23+]:

Let $X \subseteq \mathbb{C}^D$ be nice*, $W \in \mathrm{Herm}(\mathbb{C}^D)$ Hermitian.

*Any conic variety

For each k, let $\mu_k = \lambda_{\max}\left(P_X^k\left(W \otimes I^{\otimes k-1}\right)P_X^k\right)$. ← $P_X^k = \mathrm{Proj}(X^k)$

Then the μ_k form a non-increasing sequence converging to $h_X(W) \coloneqq \max_{|\psi\rangle \in X} \langle\psi|W|\psi\rangle$.

Theorem/Robust Hierarchy not only holds for P_U, but for any Hermitian W!

Robust generalization:

Instead of determining whether U avoids X,

Compute $h_X(U) \coloneqq \max_{|\psi\rangle \in X} \langle\psi|P_U|\psi\rangle$

$P_U = \mathrm{Proj}(U)$

U avoids $X \iff h_X(U) < 1$

Extending symmetric extensions to other notions of separability

Theorem/Robust Hierarchy [JLV 23+]:

Let $X \subseteq \mathbb{C}^D$ be nice*, $W \in \mathrm{Herm}(\mathbb{C}^D)$ Hermitian.

*Any conic variety

For each k, let $\mu_k = \lambda_{\max}\left(P_X^k\left(W \otimes I^{\otimes k-1}\right)P_X^k\right)$. ← $P_X^k = \mathrm{Proj}(X^k)$

Then the μ_k form a non-increasing sequence converging to $h_X(W) \coloneqq \max\limits_{|\psi\rangle \in X} \langle\psi|W|\psi\rangle$.

Definition: A density operator $\rho \in \mathrm{D}(\mathbb{C}^D)$ is X-arable if there exist

$|\psi_1\rangle, \ldots, |\psi_\ell\rangle \in X$ such that $\rho = \sum_{i=1}^{\ell} p_i |\psi_i\rangle\langle\psi_i|$.

Example: $\rho \in \mathrm{D}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is X_{Sep}-arable $\Leftrightarrow$ ρ is separable.

Theorem/Robust Hierarchy [JLV 23+]:

Let $X \subseteq \mathbb{C}^D$ be nice*, $W \in \mathrm{Herm}(\mathbb{C}^D)$ Hermitian.

*Any conic variety

For each k, let $\mu_k = \lambda_{\max}\left(P_X^k\left(W \otimes I^{\otimes k-1}\right)P_X^k\right)$. ← $P_X^k = \mathrm{Proj}(X^k)$

Then the μ_k form a non-increasing sequence converging to $h_X(W) \coloneqq \max_{|\psi\rangle \in X} \langle\psi|W|\psi\rangle$.

Definition: A density operator $\rho \in \mathrm{D}(\mathbb{C}^D)$ is X-arable if there exist

$|\psi_1\rangle, \ldots, |\psi_\ell\rangle \in X$ such that $\rho = \sum_{i=1}^{\ell} p_i |\psi_i\rangle\langle\psi_i|$.

Corollary [JLV 23+]: ρ is X-arable $\Leftrightarrow$ for all k there exists $\sigma \in \mathrm{D}\left((\mathbb{C}^D)^{\otimes k}\right)$ such that $\mathrm{Tr}_{2,3,\ldots,k}(\sigma) = \rho$ and $\mathrm{Im}(\sigma) \subseteq X^k = \mathrm{Span}\{|\psi\rangle^{\otimes k} : |\psi\rangle \in X\}$

"σ is an extension of ρ"

Example (Symmetric extensions) [DPS 04]: ρ is separable $\Leftrightarrow$ For all k there exists an extension σ of ρ such that $\mathrm{Im}(\sigma) \subseteq X_{\mathrm{Sep}}^k = S^k(\mathbb{C}^d) \otimes S^k(\mathbb{C}^d)$ ← Symmetric

Theorem/Robust Hierarchy [JLV 23+]:

Let $X \subseteq \mathbb{C}^D$ be nice*, $W \in \mathrm{Herm}(\mathbb{C}^D)$ Hermitian.

*Any conic variety

For each k, let $\mu_k = \lambda_{\max}\left(P_X^k\left(W \otimes I^{\otimes k-1}\right)P_X^k\right)$. ← $P_X^k = \mathrm{Proj}(X^k)$

Then the μ_k form a non-increasing sequence converging to $h_X(W) := \max\limits_{|\psi\rangle \in X} \langle\psi|W|\psi\rangle$.

Definition: A density operator $\rho \in \mathrm{D}(\mathbb{C}^D)$ is X-arable if there exist

$|\psi_1\rangle, \ldots, |\psi_\ell\rangle \in X$ such that $\rho = \sum_{i=1}^{\ell} p_i |\psi_i\rangle\langle\psi_i|$.

Corollary [JLV 23+]: ρ is X-arable $\iff$ for all k there exists $\sigma \in \mathrm{D}\left((\mathbb{C}^D)^{\otimes k}\right)$

such that $\mathrm{Tr}_{2,3,\ldots,k}(\sigma) = \rho$ and $\mathrm{Im}(\sigma) \subseteq X^k = \mathrm{Span}\{|\psi\rangle^{\otimes k} : |\psi\rangle \in X\}$

"σ is an extension of ρ"

"in-X-arability witness"

Proof uses robust hierarchy and the Hyperplane separation theorem:

ρ is X-arable $\iff$ $\mathrm{Tr}(\rho W) \leq 0$ for all W such that $h_X(W) \leq 0$

Examples

Theorem [JLV 23+]: ρ X-arable $\Leftrightarrow$ $\forall k\ \exists$ extension σ of ρ s.t. $\mathrm{Im}(\sigma) \subseteq X^k = \mathrm{Span}\{|\psi\rangle^{\otimes k} : |\psi\rangle \in X\}$

Schmidt rank $\leq r$ states [TMG 15]

$X_r = \{|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d : \mathrm{SR}(|\psi\rangle) \leq r\}$

ρ X_r-able $\Leftrightarrow$ $\forall k\ \exists$ extension σ of ρ s.t. $\mathrm{Im}(\sigma) \subseteq S^k(\mathbb{C}^d \otimes \mathbb{C}^d) \cap \left(\left(\wedge^{r+1}(\mathbb{C}^d)^{\otimes 2}\right)^{\perp} \otimes (\mathbb{C}^d \otimes \mathbb{C}^d)^{\otimes k-r-1}\right)$

Product states [DPS 04]

$X_{\mathrm{Sep}} = \{|\psi_1\rangle \otimes \cdots \otimes |\psi_m\rangle : |\psi_i\rangle \in \mathbb{C}^d\}$

ρ separable $\Leftrightarrow$ $\forall k\ \exists$ extension σ of ρ s.t. $\mathrm{Im}(\sigma) \subseteq S^k(\mathbb{C}^d) \otimes \cdots \otimes S^k(\mathbb{C}^d)$

Biseparable states

$X_B = \{|\psi\rangle \in (\mathbb{C}^d)^{\otimes m}$: SR($|\psi\rangle$)=1 in some cut$\}$

ρ biseparable $\Leftrightarrow$ $\forall k\ \exists$ extension σ of ρ s.t $\mathrm{Im}(\sigma) \subseteq \sum_{T \subseteq [m]} S^k\left((\mathbb{C}^d)^{\otimes T}\right) \otimes S^k\left((\mathbb{C}^d)^{\otimes [m]-T}\right)$

Slice rank 1 states

$X_S = \{|\psi\rangle \in (\mathbb{C}^n)^{\otimes m}$: SR($|\psi\rangle$)=1 in some 1 vs. rest$\}$

ρ X_S-arable $\Leftrightarrow$ $\forall k\ \exists$ extension σ of ρ s.t $\mathrm{Im}(\sigma) \subseteq \sum_{i=1}^m S^k(\mathbb{C}^d) \otimes S^k\left((\mathbb{C}^d)^{\otimes m-1}\right)$

MPS of bond dimension $\leq r$

$X_{\mathrm{MPS},\,r} = \{|\psi\rangle \in (\mathbb{C}^n)^{\otimes m}$: Every L-R cut has SR $\leq r\}$

ρ $X_{\mathrm{MPS},\,r}$-arable $\Leftrightarrow$ $\forall k\ \exists$ extension σ of ρ s.t

…

Examples

Theorem [JLV 23+]: ρ X-arable $\iff$ $\forall k$ $\exists$ extension σ of ρ s.t.
$\mathrm{Im}(\sigma) \subseteq X^k = \mathrm{Span}\{|\psi\rangle^{\otimes k} : |\psi\rangle \in X\}$

Schmidt rank $\leq r$ states [TMG 15] ρ X_r-able $\iff$ $\forall k$ $\exists$ extension σ of ρ s.t.

$X_r = \{|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d : \mathrm{SR}(|\psi\rangle) \leq r\}$ $\mathrm{Im}(\sigma) \subseteq S^k(\mathbb{C}^d \otimes \mathbb{C}^d) \cap \left(\left(\wedge^{r+1}(\mathbb{C}^d)^{\otimes 2}\right)^{\perp} \otimes (\mathbb{C}^d \otimes \mathbb{C}^d)^{\otimes k-r-1}\right)$

Product states [DPS 04] ρ separable $\iff$ $\forall k$ $\exists$ extension σ of ρ s.t.

$X_{\mathrm{Sep}} = \{|\psi$ … $(\mathbb{C}^d)$

Biseparable …

$X_B = \{|\psi\rangle \in$ … $[m]-T$ …

Takeaway:

Symmetric extensions hierarchy for separability

extended to…

Linearly constrained extensions hierarchy for X-arability

Slice rank 1 states ρ X_S-arable $\iff$ $\forall k$ $\exists$ extension σ of ρ s.t

$X_S = \{|\psi\rangle \in (\mathbb{C}^n)^{\otimes m}$: SR($|\psi\rangle$)=1 in some 1 vs. rest$\}$ $\mathrm{Im}(\sigma) \subseteq \sum_{i=1}^{m} S^k(\mathbb{C}^d) \otimes S^k\left((\mathbb{C}^d)^{\otimes m-1}\right)$

MPS of bond dimension $\leq r$ ρ $X_{\mathrm{MPS},\,r}$-arable $\iff$ $\forall k$ $\exists$ extension σ of ρ s.t

$X_{\mathrm{MPS},\,r} = \{|\psi\rangle \in (\mathbb{C}^n)^{\otimes m}$: Every L-R cut has SR $\leq r\}$ …

Conclusion

1. **Complete hierarchies** of linear systems to certify entanglement of a subspace. These work extremely well already at early levels.

Title: Complete hierarchy of linear systems for certifying quantum entanglement of subspaces

2. **(Briefly mentioned) poly-time algorithms** to find low-entanglement elements of a subspace. These also work extremely well.

Title: Computing linear sections of varieties: quantum entanglement, tensor decompositions and beyond

3. **Extending symmetric extensions:** Separability testing hierarchy of [DPS 04] extended to hierarchies for Schmidt number, biseparability, and X-arability.

Title: TBD

A complete hierarchy of linear systems for certifying quantum entanglement of subspaces

Nathaniel Johnston[1] Benjamin Lovitz[2] Aravindan Vijayaraghavan[3]

1. Mount Allison University

2. NSF Postdoc, Northeastern University

3. Northwestern University

QIP 2023

February 6, 2023

Northeastern University