Quantum Walks on Graphs

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Outline

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2. Dirac Notation
3. Quantum Walks
4. Example: Hypercubes!
5. Quotient Graphs
6. What we've Done
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- ⋆ Perfect State Transfer
- ⋆ Quantum Algorithms
  - Factoring Algorithm (1994 Peter Shor)

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- ⋆ $|v\rangle$ denotes a column vector and $\langle u|$ denotes a row vector.

$$|v\rangle = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \langle u| = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}$$

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- When a scalar i is placed inside the Dirac brackets, it represents a vector of all zeroes except a single 1 (one) in the i^{th} entry.

$$\langle i| = \begin{bmatrix} 0_1 & \dots & 0_{i-1} & 1_i & 0_{i+1} & \dots & 0_n \end{bmatrix}$$

Definitions

- Given a graph $G = (V, E)$, we define its *adjacency matrix* $A(G)$ as

$$A_{i,j} = \langle j|A|i\rangle = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

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- We define the *Laplacian* matrix $L(G)$ as

$$L_{i,j} = \langle j|L|i\rangle = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G) \\ -d(v_j) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

More Definitions!

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- The *quantum walk* can also be set up physically when the Laplacian is used in place of the adjacency matrix.
- The probability of starting at a vertex a and ending at a vertex b at time $\tilde{t}$ is given by $\left| \langle b | U(\tilde{t}) | a \rangle \right|^2$.

- **Definition:** Given a graph G, we say there is *perfect state transfer* (PST) from vertex a to vertex b if there is a time $\tilde{t}$ such that $\left|\langle b|U(\tilde{t})|a\rangle\right|^2 = 1$.

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- We usually evaluate whether a graph exhibits PST or $PGST$ (or neither) using the **Spectral Decomposition Theorem**: Any n-vertex adjacency (or Laplacian) matrix with eigenvalues λ_k and eigenvectors v_k can be written as the sum $A = \sum_{k=0}^{n} \lambda_k |v_k\rangle\langle v_k|$

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- The *quantum walk* then becomes $U(t) = \sum_{k=0}^{n} e^{-it\lambda_k}|v_k\rangle\langle v_k|$

A sample calculation in n parts ($n \in \mathbb{N}$)

1. Why Q2?
2. Graph to Matrix
3. Matrix to Eigenstuff
4. Eigenstuff to Quantum Walk
5. Quantum Walk to PST (Or not? Spoilers!)

Hypercubes

Figure: Tesseract = Q4 = 4D hypercube

Hypercubes

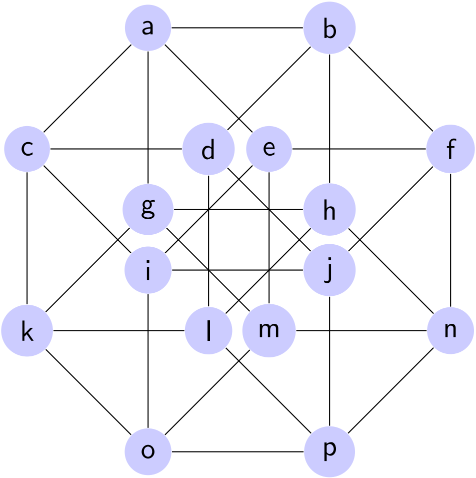

Figure: Tesseract = Q4 = 4D hypercube

Q2: Graph

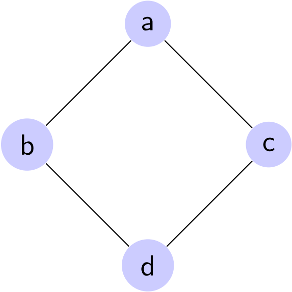

Figure: Q2

Q2: Adjacency Matrix

$$\begin{array}{cccccc} & & a & b & c & d \\ & & \downarrow & \downarrow & \downarrow & \downarrow \\ a & \rightarrow & \_ & \_ & \_ & \_ \\ b & \rightarrow & \_ & \_ & \_ & \_ \\ c & \rightarrow & \_ & \_ & \_ & \_ \\ d & \rightarrow & \_ & \_ & \_ & \_ \end{array}$$

Q2: Adjacency Matrix

$$\begin{array}{ccccc} & a & b & c & d \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ a \rightarrow & 0 & \_ & \_ & \_ \\ b \rightarrow & \_ & 0 & \_ & \_ \\ c \rightarrow & \_ & \_ & 0 & \_ \\ d \rightarrow & \_ & \_ & \_ & 0 \end{array}$$

Q2: Adjacency Matrix

$$
\begin{array}{cccccc}
 & & a & b & c & d \\
 & & \downarrow & \downarrow & \downarrow & \downarrow \\
a & \rightarrow & 0 & 1 & 1 & \_ \\
b & \rightarrow & 1 & 0 & \_ & 1 \\
c & \rightarrow & 1 & \_ & 0 & 1 \\
d & \rightarrow & \_ & 1 & 1 & 0
\end{array}
$$

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d & \rightarrow & 0 & 1 & 1 & 0
\end{array}
$$

Q2: Adjacency Matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Next up: Spectral Decomposition Theorem!

Q2: Eigenvalues

$$\begin{vmatrix} -\lambda & 1 & 1 & 0 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2(\lambda+2)(\lambda-2) = 0$$

$$\rightarrow \lambda = 0, \pm 2$$

Q2: Eigenvectors

$$\begin{pmatrix} -\lambda & 1 & 1 & 0 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 0 & 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\lambda = 0 : \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \lambda = 2 : \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \lambda = -2 : \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$e^{-itM} = \text{quantum walk on M}$$

$$\frac{1}{2}\begin{pmatrix} cos(2t)+1 & -i\ sin(2t) & -i\ sin(2t) & cos(2t)-1 \\ -i\ sin(2t) & cos(2t)+1 & cos(2t)-1 & -i\ sin(2t) \\ -i\ sin(2t) & cos(2t)-1 & cos(2t)+1 & -i\ sin(2t) \\ cos(2t)-1 & -i\ sin(2t) & -i\ sin(2t) & cos(2t)+1 \end{pmatrix}$$

Moment of Truth: PST?

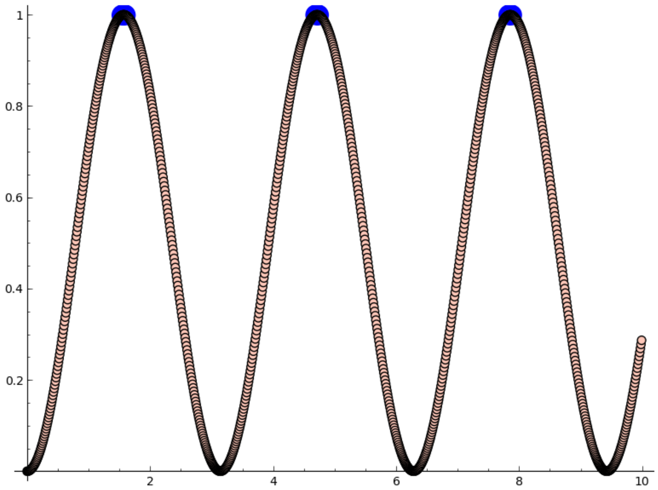

Figure: Quantum walk on Q2

Calculating PST

So we have a method! But it's:

1. Hard
2. Slow
3. Boring
4. Involves arithmetic
5. Gets exponentially worse on big graphs

We can also use programs. But they have problems too:

1. Only an approximation
2. Hard to differentiate between PST and P(retty Good)ST
3. Not a rigorous proof
4. Buggy
5. Still get exponentially worse on big graphs

So now what?

What if we could turn this...

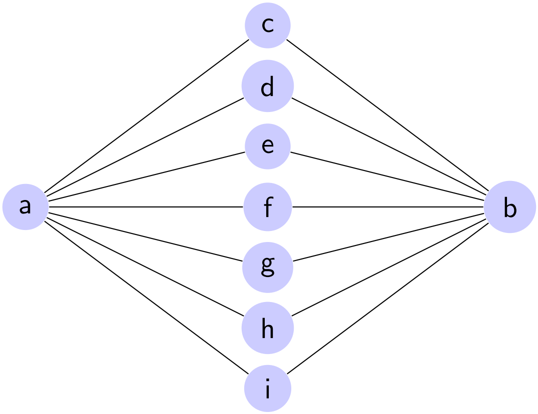

Figure: Ugly Graph

... into this?

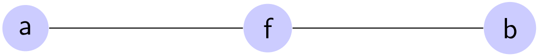

Figure: Pretty Graph

A partition is denoted by π (for obvious reasons).

Figure: Partitioning a circle

Partitions

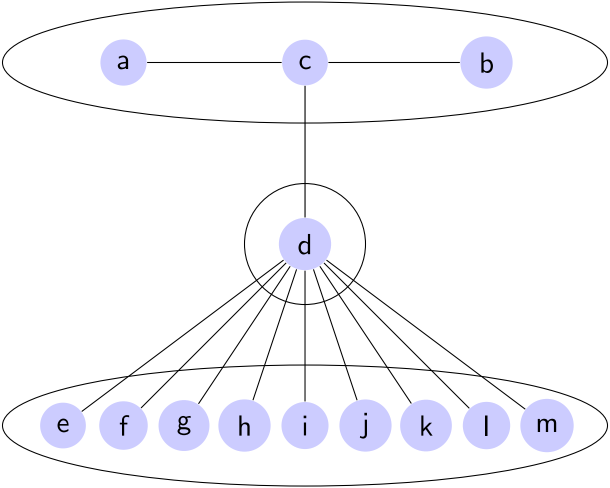

Figure: A sample partition

Almost-Equitable Partition:

$$\forall A, B \text{ partitions}, \forall a_1, a_2 \in A,$$
$$\sum_{b \in B} \begin{cases} 1 \text{ if } (a_1, b) \in E \\ 0 \text{ otherwise} \end{cases} = \sum_{b \in B} \begin{cases} 1 \text{ if } (a_2, b) \in E \\ 0 \text{ otherwise} \end{cases}$$

Translation: Any two points in the same partition must have the same number of edges going from them into any given other partition.

Equitable Partition = almost-equitable + each subgraph is regular

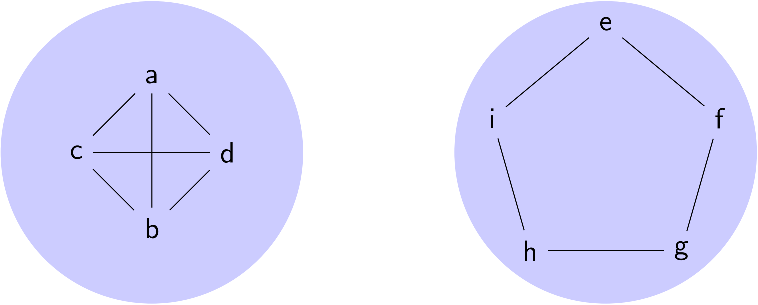

Figure: A sample partition

Almost-Equitable Partitions

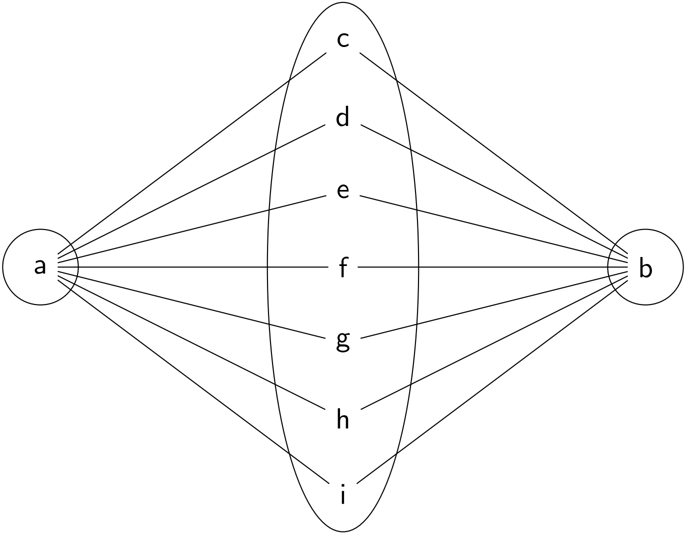

Figure: An almost-equitable partition π

Quotient Graph

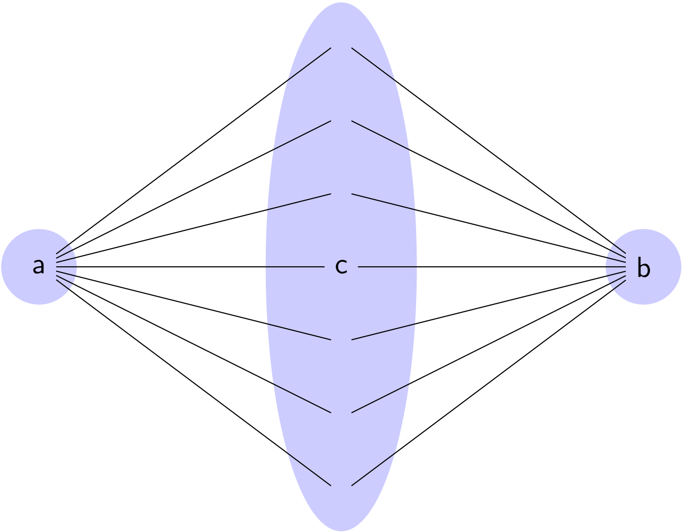

Figure: A quotient graph (almost)

Quotient Graph

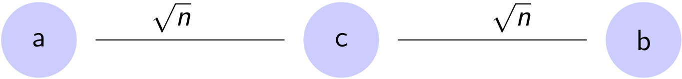

Figure: A quotient graph G/π

What We've Done

Theorem. Let H be any graph of n vertices. Let $G = \overline{K_2} + H$.
Let $\pi = \bigcup_k V_k$ be any almost equitable partition of G.
Then, G/π has PST from a to b when analyzed with its Laplacian matrix $L(G/\pi)$ iff $n \equiv 2 \bmod 4$.

Spectra of $L(G/\pi)$

The Laplacian of G, $L(G/\pi)$, is given by:

$$\begin{bmatrix} n & -\sqrt{n} & 0 \\ -\sqrt{n} & 2 & -\sqrt{n} \\ 0 & -\sqrt{n} & n \end{bmatrix}$$

The eigenvalues of $L(G/\pi)$ include $0, n, n+2$. The eigenvalue/eigenvector pairs with their respective normalizing constants are:

$$\lambda = 0, \frac{1}{\sqrt{2+n}} \begin{bmatrix} 1 \\ \sqrt{n} \\ 1 \end{bmatrix} ; \lambda = n, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} ; \lambda = n+2, \frac{1}{\sqrt{2(n+2)}} \begin{bmatrix} \sqrt{n} \\ -2 \\ \sqrt{n} \end{bmatrix}$$

Quantum Walk on $L(G/\pi)$

By the Spectral Decomposition Theorem, the quantum walk from a to b on $L(G/\pi)$ is given by:

$$\langle b|e^{-itL(G/\pi)}|a\rangle = \frac{1}{2+n} - \frac{e^{-it(n)}}{2} + \frac{ne^{-it(n+2)}}{2(2+n)} = -\frac{e^{-it(n)}}{2} + \frac{2+ne^{-it(n+2)}}{2(2+n)}.$$

$n \equiv 2 \bmod 4 \implies G/\pi$ has PST from a to b

Since $n \equiv 2 \bmod 4$, $n = 4k + 2 = 2(2k + 1)$ for some $k \in \mathbb{Z}$. Choose $t = \frac{\pi}{2}$. Then,

$$e^{-it(n)} = \cos((2k+1)\pi) + i\sin((2k+1)\pi) = -1.$$

Also,

$$e^{-it(n+2)} = \cos((2(k+1))\pi) + i\sin((2(k+1))\pi) = 1.$$

Since $e^{-it(n)} = -1$ and $e^{-it(n+2)} = 1$,

$$|-\frac{e^{-it(n)}}{2} + \frac{2 + ne^{-it(n+2)}}{2(2+n)}| = |-\frac{-1}{2} + \frac{2+n}{2(2+n)}| = 1.$$

G/π has PST from a to b $\implies$ $n \equiv 2 \bmod 4$

Since G/π has PST from a to b, for some time $t \in \mathbb{R}$,

$$|\frac{1}{2+n} - \frac{e^{-it(n)}}{2} + \frac{ne^{-it(n+2)}}{2(2+n)}| = 1.$$

Observe that $\frac{1}{2+n} = \frac{1}{2+n}e^{i(2\pi)k}$ for some $k \in \mathbb{Z}$. Then,

$$|\frac{1}{2+n}e^{i(2\pi)k} + \frac{e^{i(-(nt+\pi))}}{2} + \frac{ne^{i(-t(n+2))}}{2(2+n)}| = 1.$$

Borrowing a Lemma from REU 2013

Since $\{\frac{1}{2+n}, \frac{1}{2}, \frac{n}{2(2+n)}\}$ is a positive set of real numbers such that

$$\frac{1}{2+n} + \frac{1}{2} + \frac{n}{2(2+n)} = 1$$

and $\{(2\pi)k, -(nt+\pi), -t(n+2)\}$ is a set of real numbers such that

$$|\frac{1}{2+n}e^{i(2\pi)k} + \frac{e^{i(-(nt+\pi))}}{2} + \frac{ne^{i(-t(n+2))}}{2(2+n)}| = 1,$$

there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = (2\pi)k = -(nt+\pi) = -t(n+2).$$

Continued

Then,

$$\pi = 2t \implies 2(2t)k = -t(n+2) \implies n = -(4k+2).$$

Hence, $n \equiv 2 \mod 4$.

Recent Godsil-free Development

Theorem. Assume G is a hypercube besides $K2$, and H is an arbitrary graph on n vertices where n is a multiple of 4. Then the join $G + H$ will preserve the Laplacian PST on G with equal periodicity.

Where We're Going

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- What types of graph operations may we use on G and preserve its PST?
- Can we invent new graph operations or generalize existing graph operations to help answer this question?
- When we experiment with the type of PST which occurs (Laplacian vs. Adjacency), will our results change?

Types of Graph Operations

1. The G-Join, $G[G_1, G_2, \ldots, G_m]$
2. The Weak product
3. The Strong product
4. The Lexicographic product

G-Join

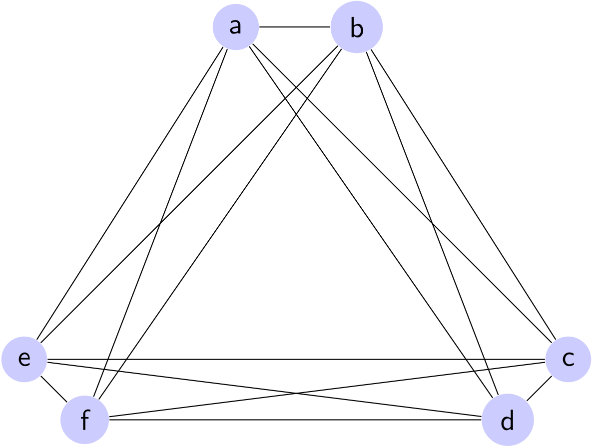

Figure: $C_3[P_2, P_2, P_2,]$

Types of Laplacian Matrices

1. The regular laplacian, $L(G) = A - D$ [Join]
2. The signless laplacian, $L(G) = A + D$ [Line Graphs]
3. The normalized laplacian, $L(G) = D^{-1/2}(A - D)D^{-1/2}$ [Weak/Strong/Lexicographic Product]

Quantum Walks on Line Graphs

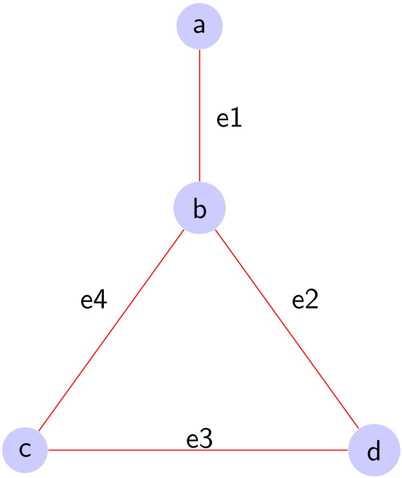

Figure: Original graph

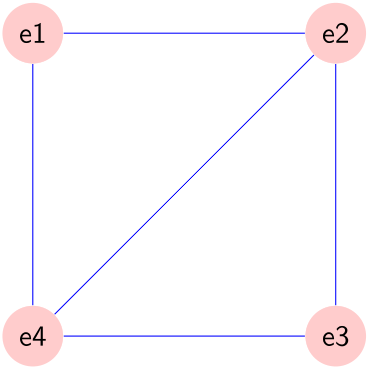

Figure: The Line Graph

There is a relationship between the quantum walk on the Line Graph and the quantum walk on the original graph.

Thanks!

1. Dr. Tamon
2. Dr. Foisy
3. Clarkson, SUNY Potsdam
4. The NSF