Tensors: Entanglement, Geometry, and Combinatorics

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Tensors and their decompositions

- Let X, Y, Z be $\mathbb{F}$-vector spaces. A tensor is an element $T \in X \otimes Y \otimes Z$
- A decomposition of T is an expression of T as a sum of product tensors

$$T = \sum_{a=1}^{n} x_a \otimes y_a \otimes z_a \, . \qquad (1)$$

- rank(T)= minimum number of product tensors needed to decompose T.
- An expression (1) is the unique rank decomposition of T if rank(T)=n and any other rank decomposition of T is the same up to re-ordering summands.

Outline

We can view tensors in three different ways:

Chapter 7
- Algebraic statistics: View a probability vector for random variables X, Y, Z as a tensor T.

Chapter 7
- Algebraic complexity theory: View a bilinear map $X \times Y \rightarrow Z^*$, e.g. matrix multiplication, as a tensor T.

Chapters 3-6
- Quantum information: View a (pure) quantum state shared by three parties X, Y, Z as a tensor T.

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Chapter 7: Algebraic Statistics

Algebraic statistics: View a probability vector for random variables X, Y, Z as a tensor T.

- Decomposition of T ⟵ Choice of latent random variable L such that

$$T = \Pr(X,Y,Z) = \sum_l \Pr(l) \Pr(X|l) \otimes \Pr(Y|l) \otimes \Pr(Z|l)$$

$$\Pr(X,Y,Z|l) = \Pr(X|l) \otimes \Pr(Y|l) \otimes \Pr(Z|l)$$

X, Y, Z independent conditioned on each outcome l for L

- Unique rank decomposition of T ⟶ Unique choice of L with this property.

[Lovitz-Petrov 21]:

- "Splitting theorem" for sets of product tensors.
- Corollary: A generalization of Kruskal's theorem.

Generalization of Kruskal’s theorem

Generalization of Kruskal’s theorem [Lovitz-Petrov 21]: A sufficient condition for

$$T = \sum_{a=1}^{n} x_a \otimes y_a \otimes z_a$$

to be the unique rank decomposition of T.

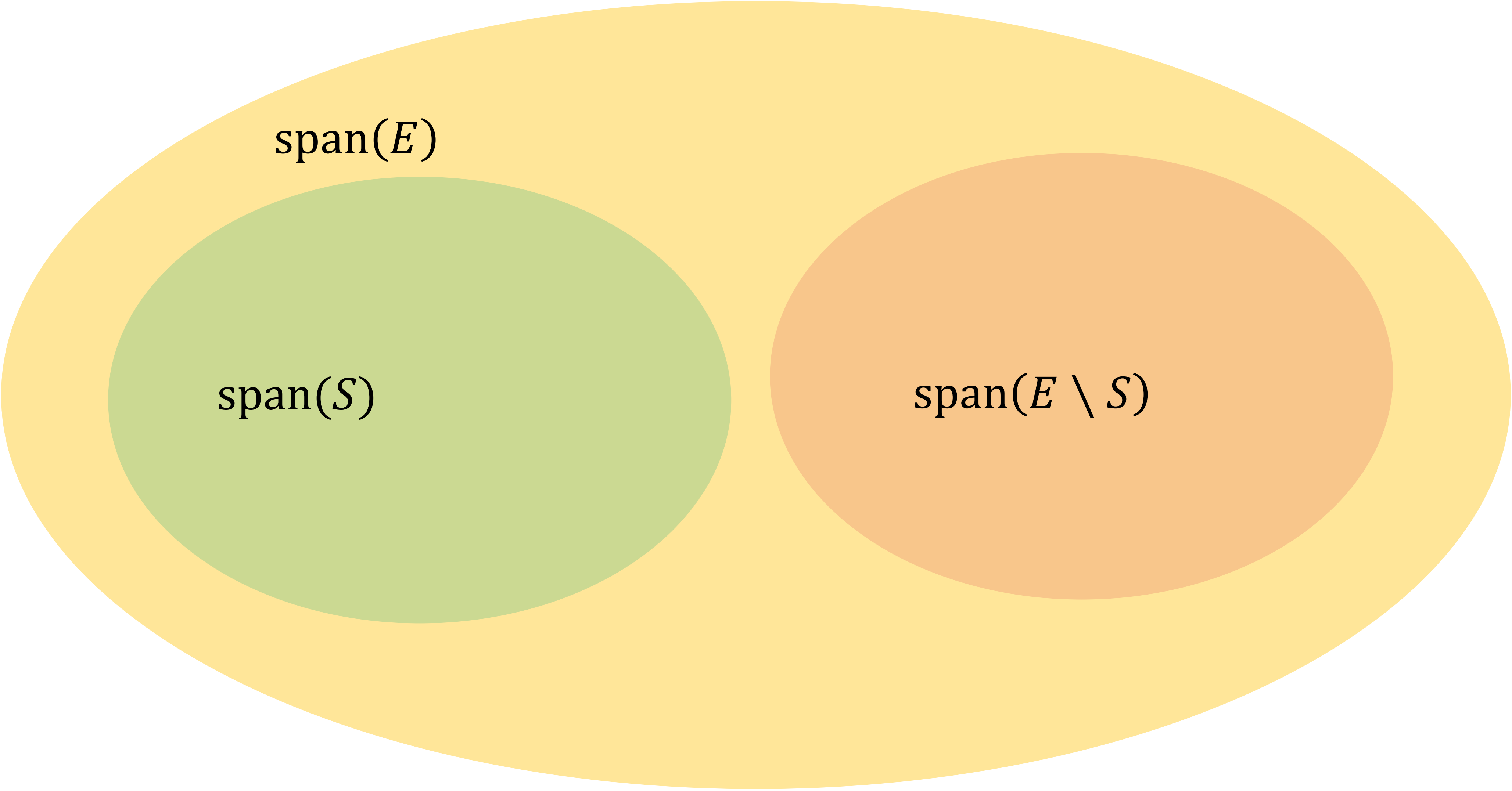
This is a corollary to our splitting theorem…

Splitting

Definition: A set of vectors $E = \{v_1, \ldots, v_n\}$ splits if there exists a non-trivial subset $S \subseteq E$ such that

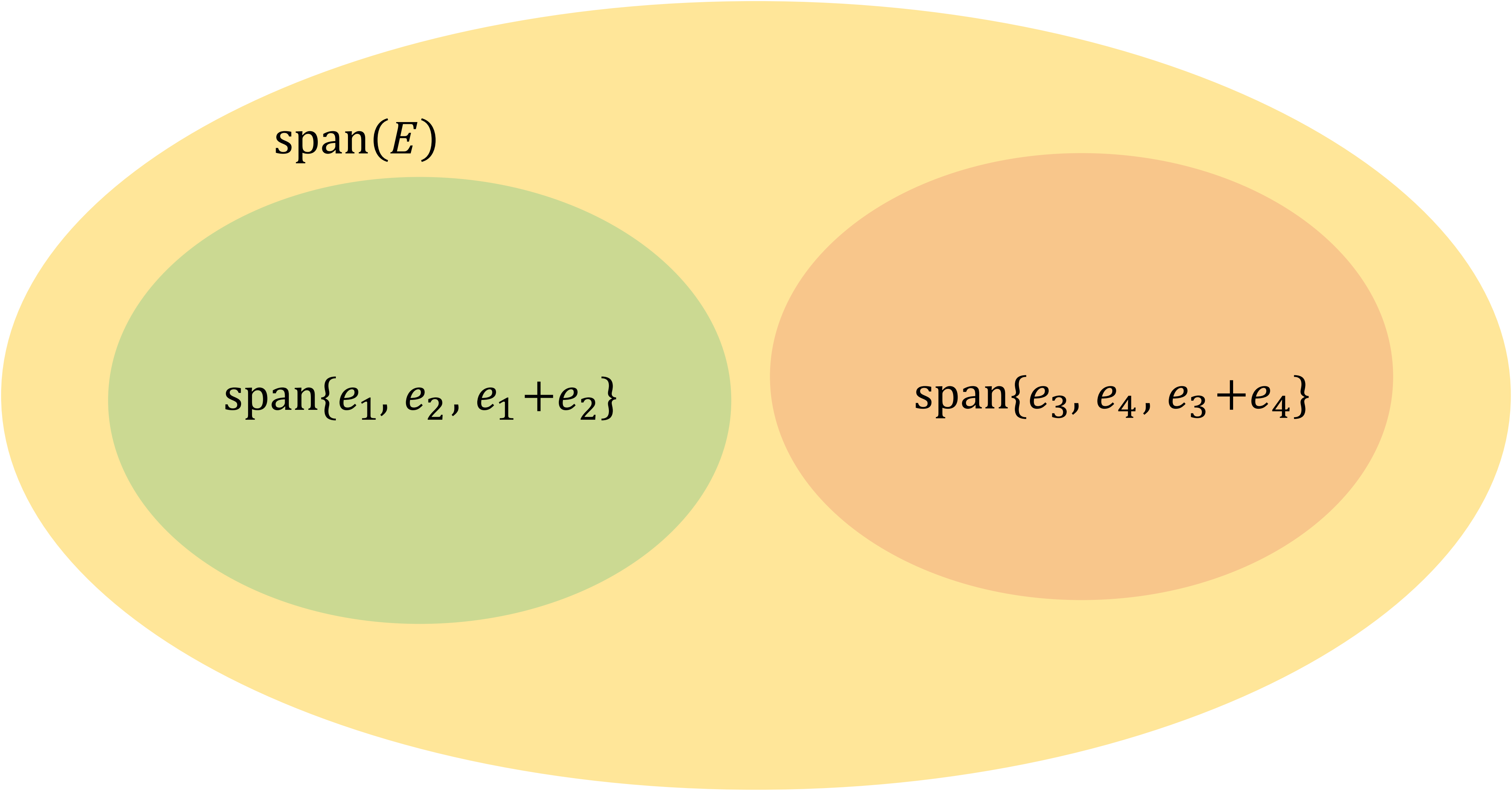
$$\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\} \quad (2)$$

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$



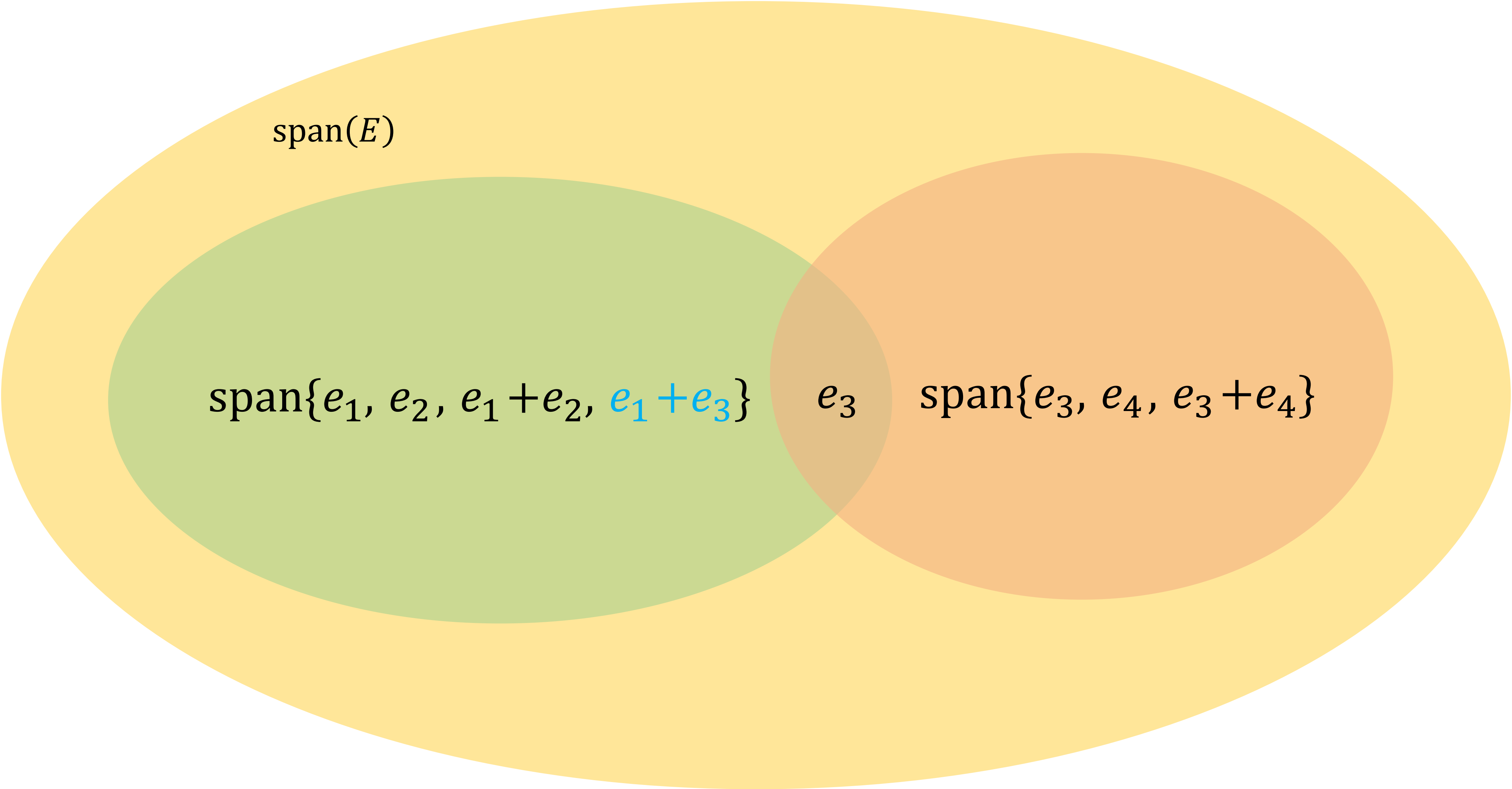
E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

$E = \{e_1, e_2, e_1+e_2, e_3, e_4, e_3+e_4\}$



E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

$$E = \{e_1, e_2, e_1+e_2, e_3, e_4, e_3+e_4\} \cup \{e_1+e_3\}$$



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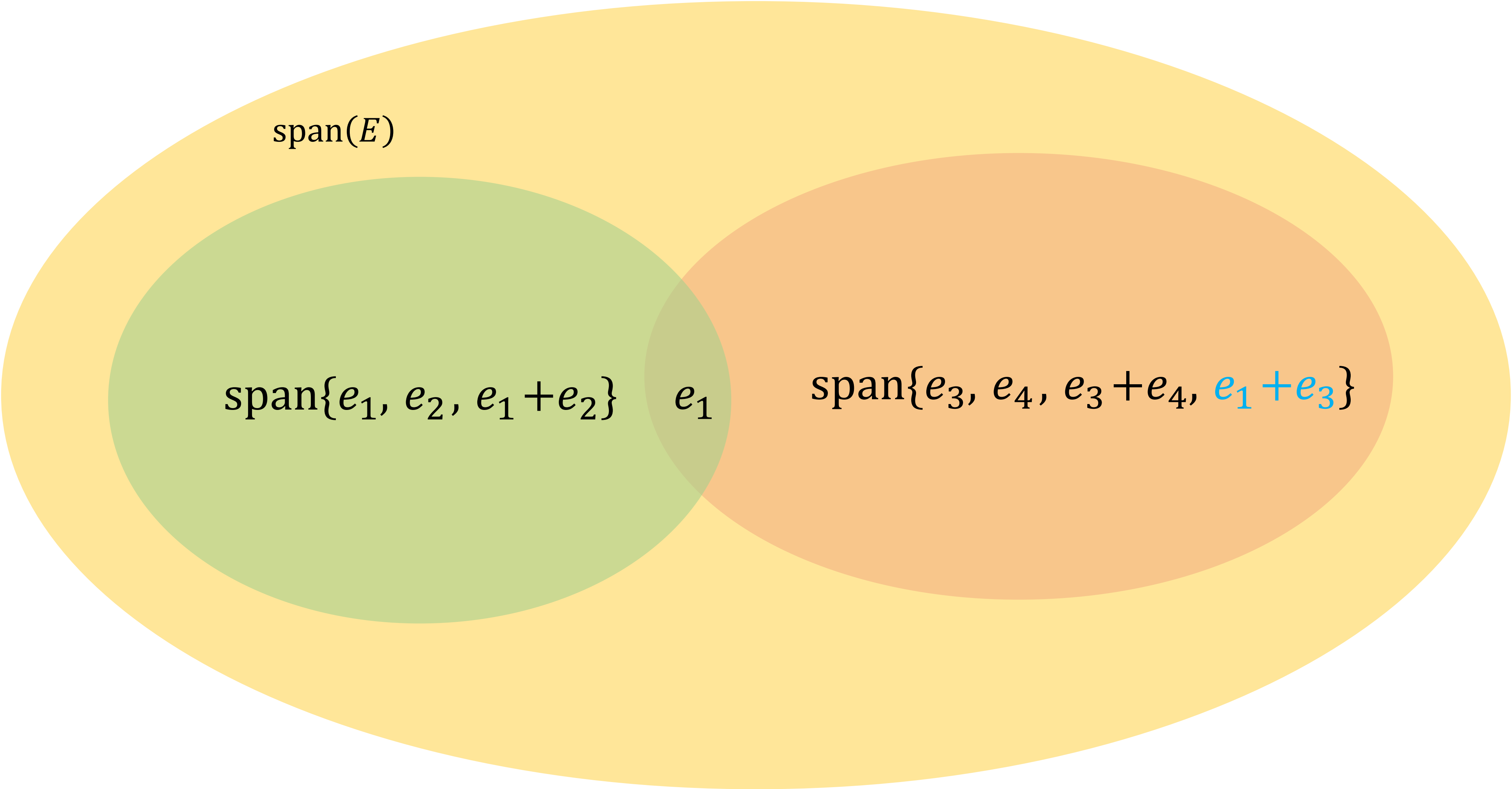

Splitting theorem

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

Splitting theorem [Lovitz-Petrov]: Let $E = \{x_a \otimes y_a : a \in [n]\}$.

If

$$\dim\mathrm{span}(E) \leq d_x^{[n]} + d_y^{[n]} - 2$$

then E splits.

$d_x^{[n]} = \dim\mathrm{span}\{x_1, \ldots, x_n\}$

(Our generalization of Kruskal's theorem is a corollary to this)

Splitting theorem

E splits if there exists $S \subseteq E$ such that $\operatorname{span}(S) \cap \operatorname{span}(E \setminus S) = \{0\}$

Splitting theorem [Lovitz-Petrov]: Let $E = \{x_a \otimes y_a \otimes z_a : a \in [n]\}$.

If

$$\operatorname{dimspan}(E) \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 3$$

then E splits.

$d_x^{[n]} = \operatorname{dimspan}\{x_1, \dots, x_n\}$

Outline

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Chapters 3-6
- Quantum information: View a (pure) quantum state shared by three parties X, Y, Z as a tensor T.

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Chapter 7: Algebraic Complexity Theory

Algebraic complexity theory: View a bilinear map $X \times Y \to Z^*$, e.g. matrix multiplication, as a tensor T.

- rank$(T) \propto$ Multiplicative complexity of applying T

[Lovitz-Petrov 21]:

- Corollary to splitting theorem: New lower bounds on tensor rank.

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Chapters 3-6: Quantum Information

Quantum information: View a (pure) quantum state shared by three parties X, Y, Z as a tensor T.

- rank(T) = "GHZ cost" of T = Smallest size of GHZ state needed to obtain T by SLOCC.
- Stabilizer rank of T = $\chi(T)$ = Computational cost of classically simulating Clifford circuits applied to T under stabilizer formalism.

Chapter 6: On Decomposable Correlation Matrices

Question: Given unit vectors $v_1, \dots, v_n \in \mathbb{C}^d$, does there exist a positive integer m and an isometry $U: \mathbb{C}^d \to (\mathbb{C}^2)^{\otimes m}$ for which $Uv_a \in (\mathbb{C}^2)^{\otimes m}$ is a product tensor for all $a \in [m]$?

If "yes": Protocol using $v_1, \dots, v_n$ could be converted into a protocol that uses tensor products of qubits.

If "no": The set $\{v_1, \dots, v_n\}$ is "absolutely entangled."

Corollary to splitting theorem: If $\{v_1, \dots, v_n\}$ does not split, then we can assume $m \leq n - 2$

Chapter 6: On Decomposable Correlation Matrices

<u>Question:</u> Given unit vectors $v_1, \dots, v_n \in \mathbb{C}^d$, does there exist a positive integer m and an isometry $U: \mathbb{C}^d \to (\mathbb{C}^2)^{\otimes m}$ for which $Uv_a \in (\mathbb{C}^2)^{\otimes m}$ is a product tensor for all $a \in [m]$?

<u>Theorem [Lovitz, 2019]:</u>

n	1	2	3	4	≥ 5
Does such a U always exist?	Yes	Yes	Yes	?	No

Chapter 3: Generic local state discrimination with pre-shared entanglement

- Question: Given a pure state $T \in X \otimes Y \otimes Z$, how useful is it for distributed computation?

- [Lovitz-Johnston 21]: For local unambiguous state discrimination,

$$\dim((GL_X \times GL_Y \times GL_Z)T)$$

  is the number of generic pure states in $X \otimes Y \otimes Z$ that can be unambiguously discriminated with pre-shared entanglement T.

- Corollary: Generic pure states T maximize this number.

Chapter 5: New techniques for bounding stabilizer rank

A state $S \in (\mathbb{C}^2)^{\otimes m}$ is a stabilizer state if $S = Ue_0^{\otimes m}$ for some Clifford circuit U.

Recall: A Clifford circuit is a circuit composed of Clifford gates

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

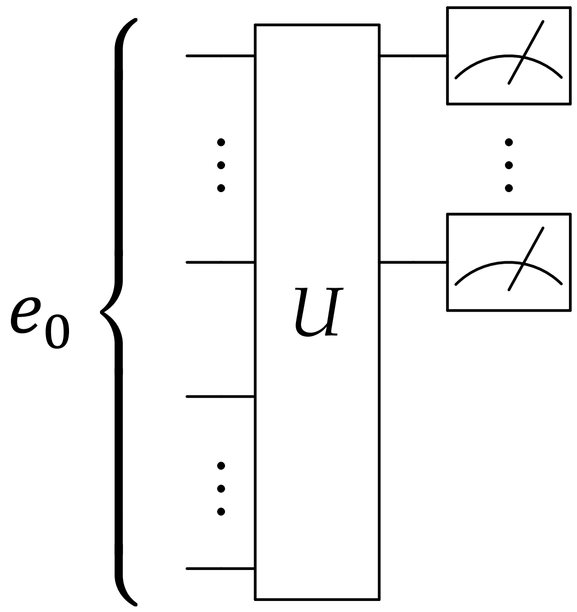
Stabilizer rank

- The stabilizer rank of a state
$$\mathrm{T} \in (\mathbb{C}^2)^{\otimes m},$$
denoted $\chi(T)$, is the minimum number r for which
$$T = \sum_{i=1}^{r} c_i S_i$$
for some $c_i \in \mathbb{C}$ and S_i stabilizer states.

- The δ-approximate stabilizer rank of T is
$$\chi_\delta(T) = \min\{\chi(R): \|T - R\| \leq \delta\}.$$

Motivation: Classical simulation

Question: Given a quantum circuit



Can it be simulated efficiently on a classical computer?

Motivation: Classical simulation

Partial answer: Write U as a sequence of Clifford+T gates (universal gate set).

Clifford gates: $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$, $CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

T gate: $\mathrm{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

[Gottesman-Knill 98]: U Clifford $\Rightarrow$ Yes.

$|H\rangle = e_0 + e^{i\pi/4} e_1$

[Bravyi-Smith-Smolin 16, Bravyi-Gosset 16]: If U consists of m T gates, then circuit can be strongly simulated with cost quadratic in $\chi\left(|H\rangle^{\otimes m}\right)$.

… and can be weakly simulated with cost quadratic in $\chi_\delta\left(|H\rangle^{\otimes m}\right)$.

Chapter 5: New techniques for bounding stabilizer rank

$\chi(|H\rangle^{\otimes m})$

- [Huang-Newman-Szegedy 20]: $\chi(|H\rangle^{\otimes m})$ super-polynomial unless P=NP.
- [Bravyi-Smith-Smolin 16]: $\chi(|H\rangle^{\otimes m}) \geq \Omega(\sqrt{m})$.
- [Peleg-Shpilka-Volk 21]: $\chi(|H\rangle^{\otimes m}) \geq \Omega(\mathrm{m})$.
- [Qassim-Pashayan-Gosset 21]: $\chi(|H\rangle^{\otimes m}) \leq 2^{\alpha m}$, where $\alpha = \frac{1}{4}\log_2(3)$.

$\chi_\delta(|H\rangle^{\otimes m})$

[Lovitz-Steffan 21]: Alternate (simplified) proofs up to log factor

- [Peleg-Shpilka-Volk 21]:

  There exists $\delta > 0$ such that $\chi_\delta(|H\rangle^{\otimes m}) \geq \Omega(\sqrt{m}/\log m)$
- [Bravyi-Gosset 16]: $\chi_\delta(|H\rangle^{\otimes m}) \leq O\left(\frac{1}{\delta^2} 2^{\alpha m}\right)$, where $\alpha \approx 0.228$.

Summary

- Algebraic statistics: View a probability vector for random variables X, Y, Z as a tensor T.
  - Chapter 7 • Splitting theorem ⇒ Generalization of Kruskal's theorem.
- Algebraic complexity theory: View a bilinear map $X \times Y \to Z^*$, e.g. matrix multiplication, as a tensor T.
  - Chapter 7 • Splitting theorem ⇒ New lower bounds on tensor rank.
- Quantum information: View a (pure) quantum state shared by three parties X, Y, Z as a tensor T.
  - Chapter 6 • Decomposable correlation matrices
  - Chapter 3 • Generic local state discrimination with pre-shared entanglement
  - Chapter 5 • New techniques for bounding stabilizer rank

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