Algorithms and Uniqueness of Tensor Decompositions

Benjamin Lovitz

Northeastern University

UMass Boston Department of Mathematics

November 14, 2023

What is a matrix?

What is a matrix?

A matrix is an element of $\mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y}$ $\begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix} \in \mathbb{F}^2 \otimes \mathbb{F}^2$

A rank-one matrix is a matrix of the form $x \otimes y = xy^T = \left(x_i y_j\right)_{(i,j)}$

What is a matrix?

A matrix is an element of $\mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y}$ $\begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix} = \binom{1}{3}(2\ 4)$

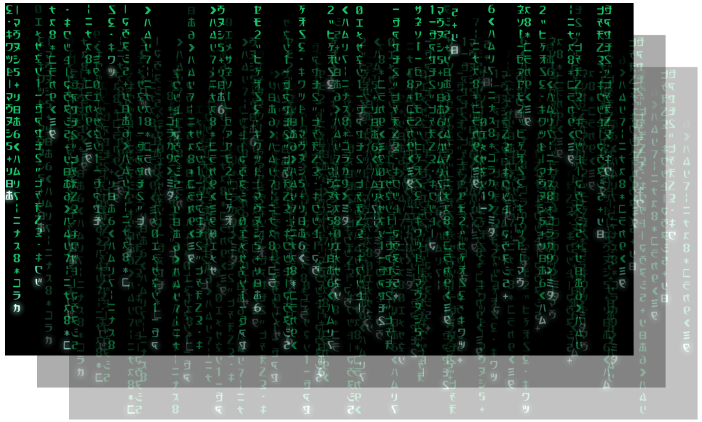
A rank-one matrix is a matrix of the form $x \otimes y = xy^T = (x_i y_j)_{(i,j)}$

A matrix is an element of $\mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y}$

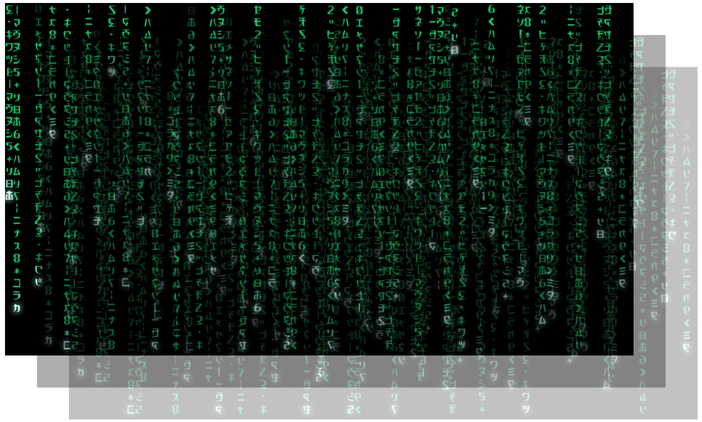
$$\begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix} = \binom{1}{3}(2\ 4)$$

A rank-one matrix is a matrix of the form $x \otimes y = xy^T = (x_i y_j)_{(i,j)}$

What is a ~~matrix~~ tensor?



What is a ~~matrix~~ tensor?

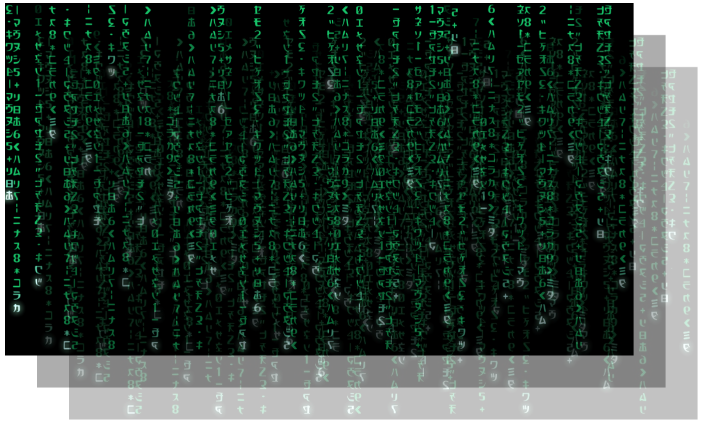


A ~~matrix~~ tensor is an element of $\mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z}$

$$\begin{bmatrix} 15 & 18 \\ 20 & 24 \end{bmatrix} \begin{bmatrix} 30 & 36 \\ 40 & 48 \end{bmatrix} \in \mathbb{F}^2 \otimes \mathbb{F}^2 \otimes \mathbb{F}^2$$

A product tensor is a tensor of the form $x \otimes y \otimes z = \left(x_i y_j z_k\right)_{(i,j,k)}$

What is a ~~matrix~~ tensor?

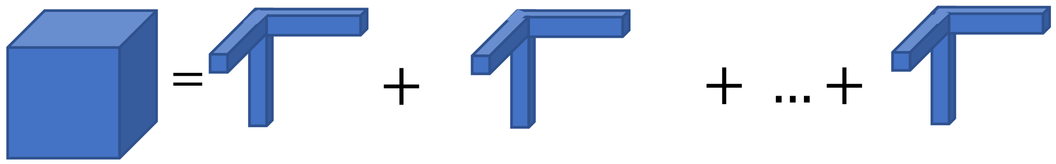


A ~~matrix~~ tensor is an element of $\mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z}$

$$\left[\begin{bmatrix} 15 & 18 \\ 20 & 24 \end{bmatrix} \begin{bmatrix} 30 & 36 \\ 40 & 48 \end{bmatrix}\right] = \binom{1}{2} \otimes \binom{3}{4} \otimes \binom{5}{6}$$

A product tensor is a tensor of the form $x \otimes y \otimes z = \left(x_i y_j z_k\right)_{(i,j,k)}$

Tensor decompositions



<u>Definition:</u> Let $n \in \mathbb{N}$ and $[n] := \{1, \dots, n\}$.

For $T \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z}$, an expression

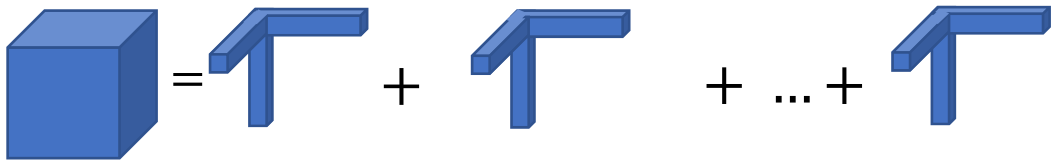
$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z}$$

is called a decomposition of T into product tensors

$\mathrm{rank}(T) :=$ smallest n

Uniqueness of tensor decompositions

<u>Definition:</u> Let $n \in \mathbb{N}$ and $[n] := \{1, \ldots, n\}$.



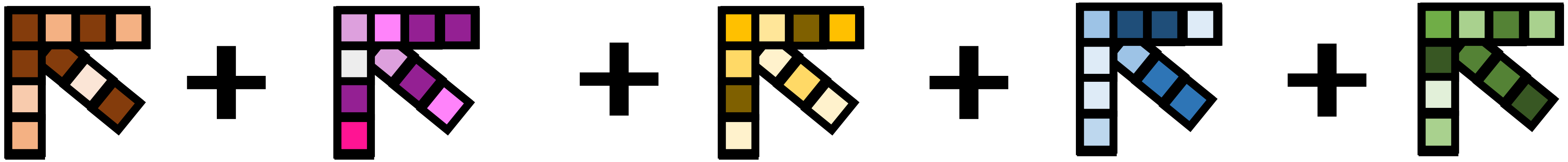
A rank decomposition

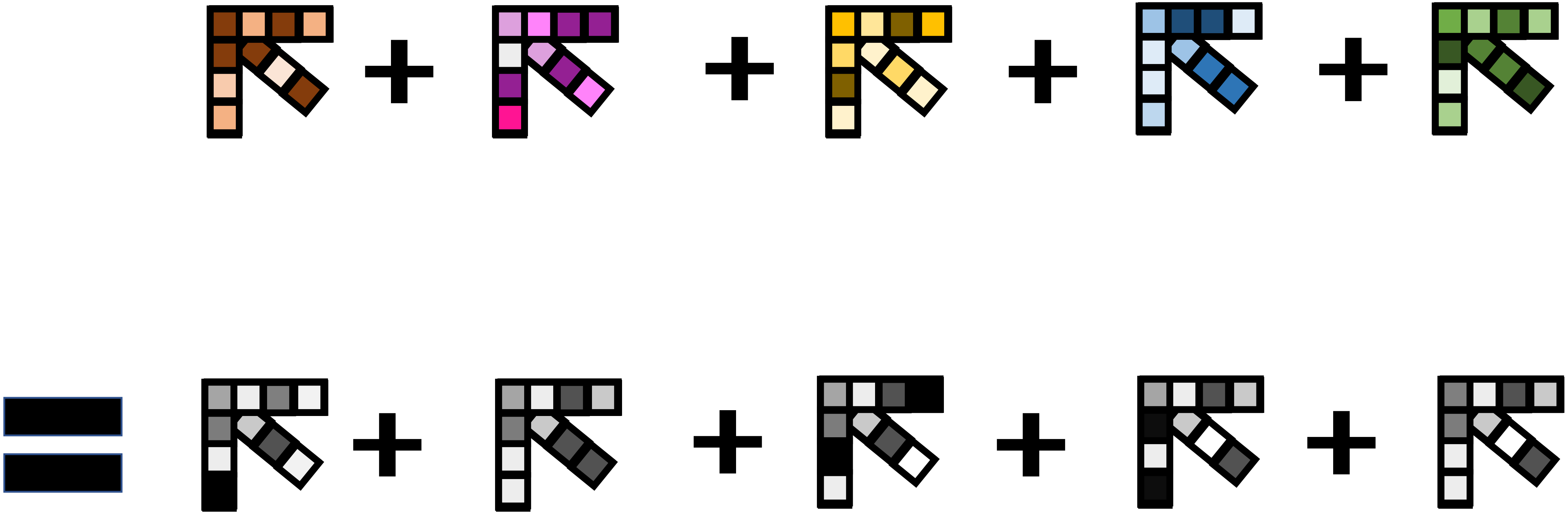
$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z}$$

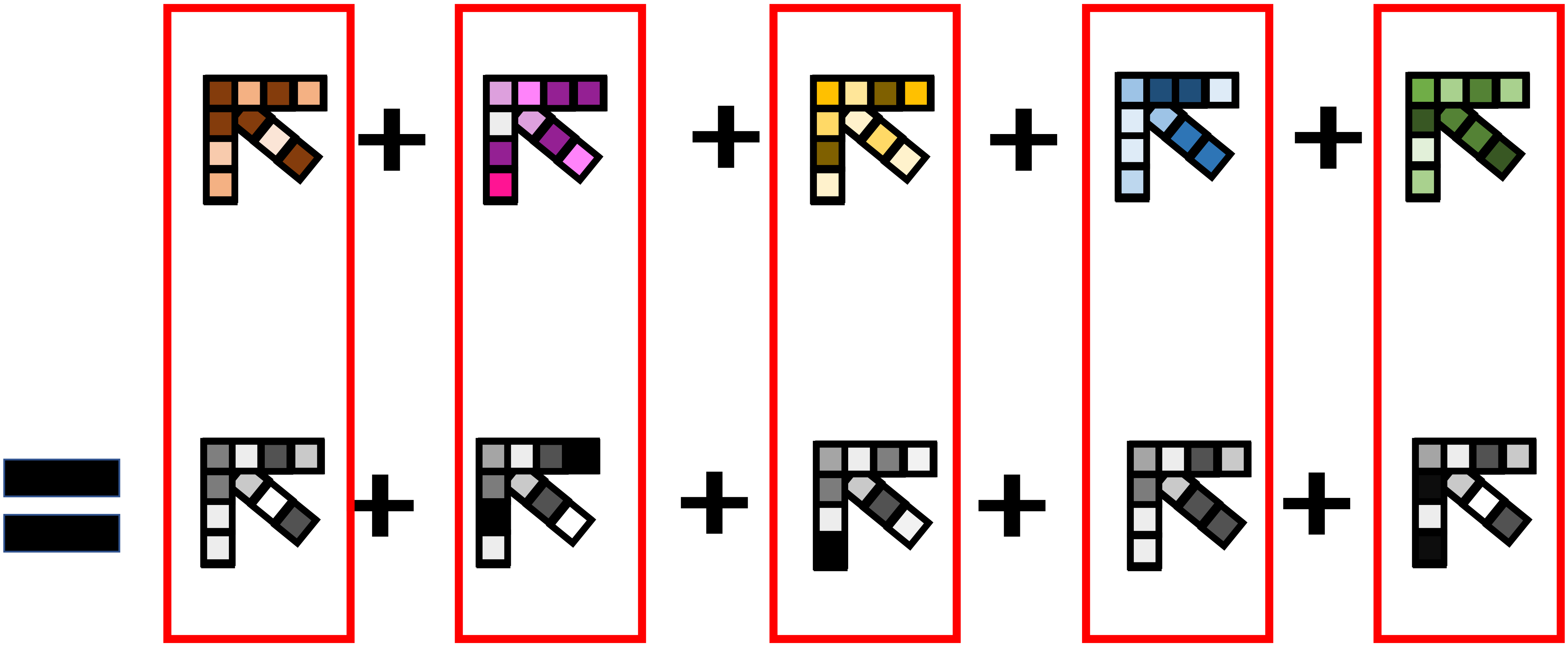
is called the unique (rank) decomposition of T if for any other decomposition

$$T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z}$$

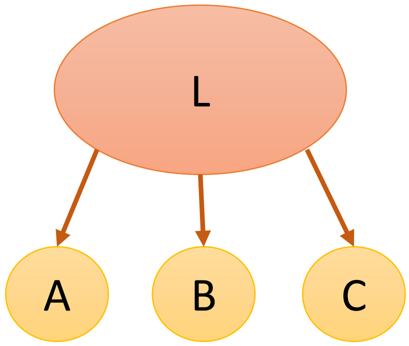
there is a permutation $\sigma \in S_n$ such that $x_a \otimes y_a \otimes z_a = x'_{\sigma(a)} \otimes y'_{\sigma(a)} \otimes z'_{\sigma(a)}$ for all $a \in [n]$.







Application: Latent parameter learning



L is for *latent*

- Let A, B, C, L be finite random variables such that A, B, C are conditionally independent, i.e.

$$\Pr(a, b, c|l) = \Pr(a|l)\Pr(b|l)\Pr(c|l) \qquad \text{for all } a, b, c, l.$$

- <u>Goal:</u> Given the probability vector $\Pr(A, B, C)$, determine $\Pr(A, B, C, L)$.
- <u>Method:</u>

$$\Pr(A, B, C) = \sum_l \Pr(l)\Pr(A, B, C|l) = \sum_l \underbrace{\Pr(l)\Pr(A|l) \otimes \Pr(B|l) \otimes \Pr(C|l)}$$

... If $\Pr(A, B, C)$ has a unique decomposition, then we can recover $\Pr(A, B, C, l)$,

- <u>Applications:</u> Learning mixtures of spherical gaussians, phylogenetic tree reconstruction, hidden Markov models, orbit retrieval, blind signal separation, document topic models, …

Two goals

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

1. **Algorithms** [JLV 2023, published in FOCS]

   Given a tensor $T \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z}$, find a rank decomposition (1).

2. **Uniqueness** [LP 2023, published in FoM Sigma]

   Given a rank decomposition (1), prove that it is the unique rank decomposition.

Two goals

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

1. **Algorithms** [JLV 2023, published in FOCS]

   Given a tensor $T \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z}$, find a rank decomposition (1).

2. **Uniqueness** [LP 2023, published in FoM Sigma]

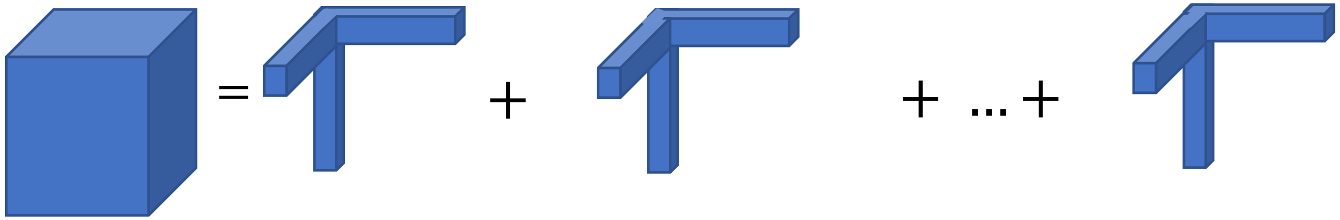
   Given a rank decomposition (1), prove that it is the unique rank decomposition.

Algorithm idea

Tensor: $T \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^k$

Decomposition: Sum of R product tensors

$$T = x_1 \otimes y_1 \otimes z_1 + x_2 \otimes y_2 \otimes z_2 + \cdots + x_n \otimes y_n \otimes z_n$$



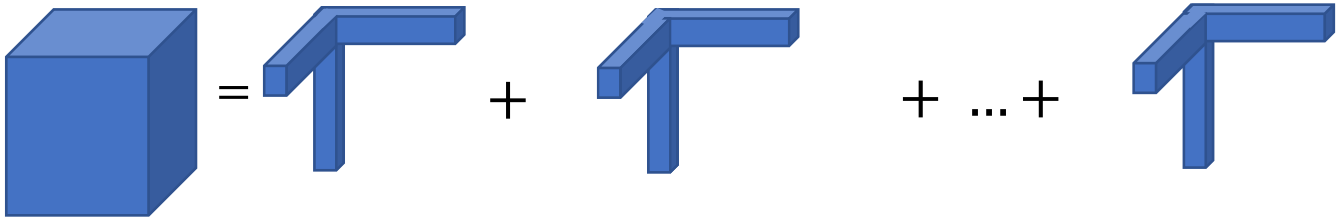
Idea: If we view T as an $d^2 \times k$ matrix, then the image is in the span of the $x_i \otimes y_i$.

Algorithm idea

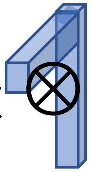
Tensor: $T \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^k$

Decomposition: Sum of R product tensors

$$T = x_1 \otimes y_1 \otimes z_1 + x_2 \otimes y_2 \otimes z_2 + \cdots + x_n \otimes y_n \otimes z_n$$



Idea: If we view T as an $d^2 \times k$ matrix, then the image is in the span of the $x_i \otimes y_i$.



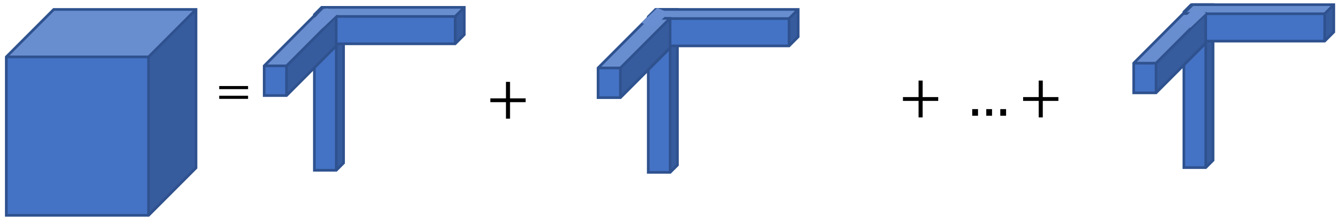
- Finding rank-one matrices in $\mathrm{im}(T)$ $\leftrightarrow$ Finding tensor decompositions of T

Algorithm idea

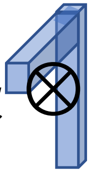
Tensor: $T \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^k$

Decomposition: Sum of R product tensors

$$T = x_1 \otimes y_1 \otimes z_1 + x_2 \otimes y_2 \otimes z_2 + \cdots + x_n \otimes y_n \otimes z_n$$



Idea: If we view T as an $d^2 \times k$ matrix, then the image is in the span of the $x_i \otimes y_i$.



- Finding rank-one matrices in $\mathrm{im}(T)$ $\leftrightarrow$ Finding tensor decompositions of T
- Finding other types of matrices in $\mathrm{im}(T)$ $\leftrightarrow$ Finding other types of decompositions of T

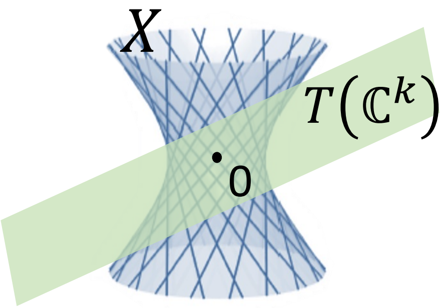
(X, k)-decompositions

For $T \in \mathbb{F}^D \otimes \mathbb{F}^k$, $X \subseteq \mathbb{C}^D$,

an (X, k)-decomposition is an expression $$T = \sum_{i=1}^{n} v_i \otimes z_i \in \mathbb{F}^D \otimes \mathbb{F}^k$$

where $v_1, \ldots, v_R \in X$

Example: When $X = X_1 = \{\text{rank 1 matrices}\} \subseteq \mathbb{F}^d \otimes \mathbb{F}^d$, an (X, k)-decomposition is just a tensor decomposition.



Viewing T as a map $\mathbb{F}^k \to \mathbb{F}^D$, each $v_i \in T(\mathbb{F}^k) \cap X$,

so computing $T(\mathbb{F}^k) \cap X \leftrightarrow (X, k)$-decomposing T

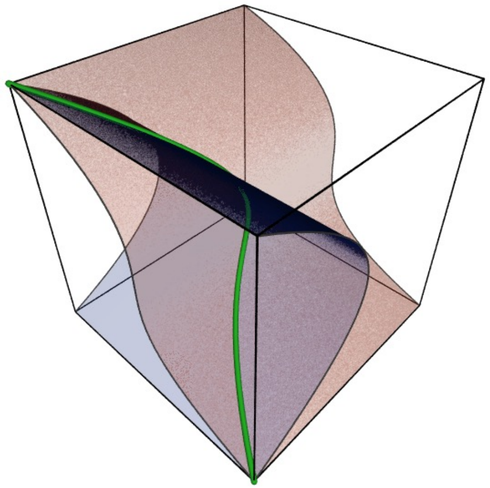
Theorem (informal) [JLV 2023]: For many algebraic varieties X,
we can recover low-rank (X, k)-decompositions efficiently.

Algebraic Varieties

Variety: common zeroes of a set of polynomials

$$X = \{x \in \mathbb{F}^D : f_1(x) = \cdots = f_p(x) = 0\}$$

- $f_1, f_2, \dots, f_p$ cut out the variety X

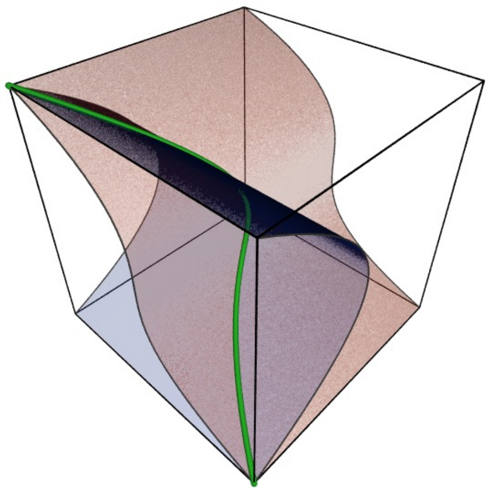


Algebraic Varieties

Variety: common zeroes of a set of polynomials

$$X = \{x \in \mathbb{F}^D : f_1(x) = \cdots = f_p(x) = 0\}$$

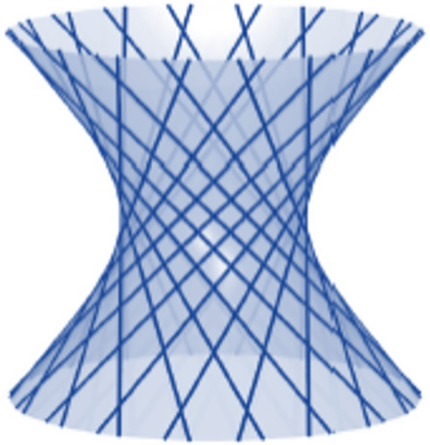
- $f_1, f_2, \ldots, f_p$ cut out the variety X



$X \subseteq \mathbb{F}^D$ is a (conic) variety iff

$$v \in X \implies \forall \lambda \in \mathbb{F}, \lambda v \in X$$

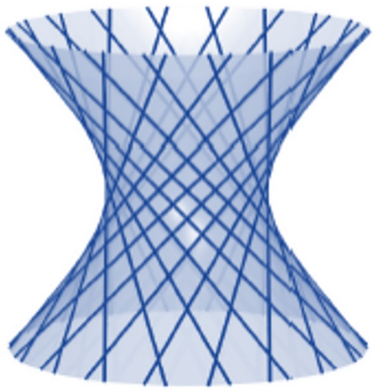
- Conic variety: $f_1, f_2, \ldots, f_p$ can be homogenous of same degree ℓ



Running example: rank-1 matrices

$$X_1 = \{u_1 \otimes u_2 \,|\, u_1 \in \mathbb{F}^{d_1}, u_2 \in \mathbb{F}^{d_2}\}$$

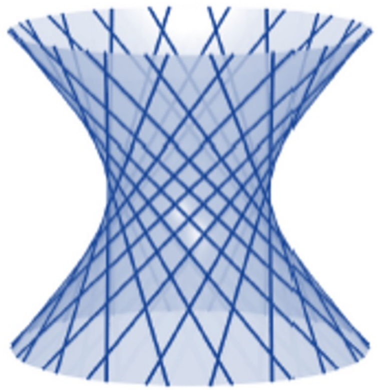
$u_1 \otimes u_2 = u_1 u_2^T$ is vector outer product.



Running example: rank-1 matrices

$$X_1 = \{u_1 \otimes u_2 \mid u_1 \in \mathbb{F}^{d_1}, u_2 \in \mathbb{F}^{d_2}\}$$

$u_1 \otimes u_2 = u_1 u_2^T$ is vector outer product.

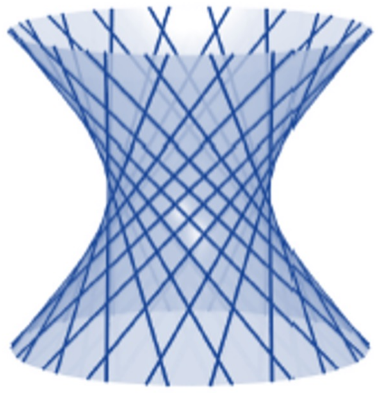


- $X_1 \subset \mathbb{F}^{d_1 \times d_2}$ is a conic variety cut out by degree-2 polynomials

Running example: rank-1 matrices

$$X_1 = \{u_1 \otimes u_2 \,|\, u_1 \in \mathbb{F}^{d_1}, u_2 \in \mathbb{F}^{d_2}\}$$

$u_1 \otimes u_2 = u_1 u_2^T$ is vector outer product.

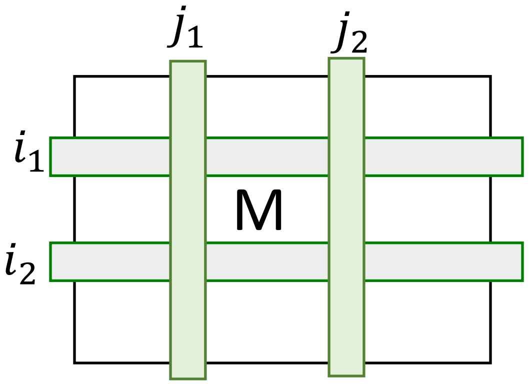


- $X_1 \subset \mathbb{F}^{d_1 \times d_2}$ is a conic variety cut out by degree-2 polynomials

$M \in X_1$ iff

$\forall 1 \leq i_1 < j_1 \leq n_1,$
$\forall 1 \leq i_2 < j_2 \leq n_2,$

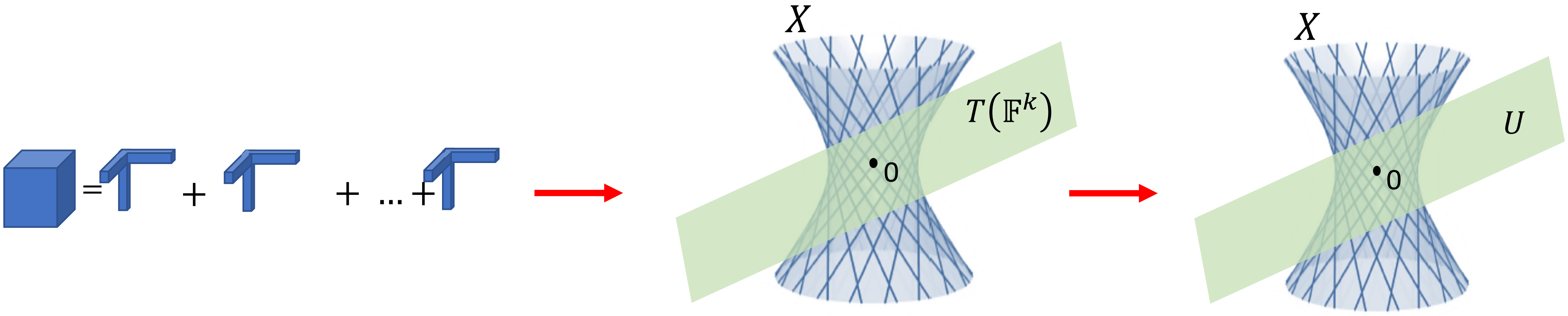
$$M_{i_1 j_1} M_{i_2 j_2} - M_{i_1 j_2} M_{i_2 j_1} = 0$$



$\det(2\ x\ 2\ submatrix) = 0$

- X_1 cut out by $\binom{n_1}{2}\binom{n_2}{2}$ homogenous degree 2 polynomials

Reduction: computing (X, k)-decompositions → computing linear sections



Algorithm for (X, k)-decompositions → Algorithm to compute $U \cap X$ for linear subspace $U = T(\mathbb{F}^k)$

Suppose $U \subseteq \mathbb{F}^D$ has a basis $\{v_1, \ldots, v_n\}$ such that each $v_i \in X$.

Problem: Given some other basis $\{u_1, \ldots, u_n\}$ of U, recover $\{v_1, \ldots, v_n\}$ (up to scale).

Example: Jennrich's Algorithm: If $U' = \text{span}\,\{v_1^{\otimes \ell}, \ldots, v_n^{\otimes \ell}\}$ with $\{v_1, \ldots, v_n\}$ linearly independent, then $\{v_1^{\otimes \ell}, \ldots, v_n^{\otimes \ell}\}$ can be recovered from any basis of U' in $D^{O(\ell)}$-time.

Suppose $U \subseteq \mathbb{F}^D$ has a basis $\{v_1, \dots, v_n\}$ such that each $v_i \in X$.

Problem: Given some other basis $\{u_1, \dots, u_n\}$ of U, recover $\{v_1, \dots, v_n\}$ (up to scale).

Example: Jennrich's Algorithm: If $U' = \text{span}\,\{v_1^{\otimes \ell}, \dots, v_n^{\otimes \ell}\}$ with $\{v_1, \dots, v_n\}$ linearly independent, then $\{v_1^{\otimes \ell}, \dots, v_n^{\otimes \ell}\}$ can be recovered from any basis of U' in $D^{O(\ell)}$-time.

Jennrich's Algorithm:

Pick $T_j \in U'$, $j = 1,2$ at random, view these as maps $T_j : (\mathbb{F}^D)^{\otimes \ell - 1} \to \mathbb{F}^D$

$T_j = \sum_{i=1}^n \alpha_{j,i} v_i \left(v_i^t\right)^{\otimes \ell - 1}$ $T_j^{-1} = \sum_i \frac{1}{\alpha_{j,i}} (w_i)^{\otimes \ell - 1} w_i^t$ where $w_i^t v_{i'} = \delta_{i,i'}$

So $T_1 T_2^{-1} = \sum_i \frac{\alpha_{1,i}}{\alpha_{2,i}} v_i w_i^t$. E-vectors / E-values of $T_1 T_2^{-1}$ are v_i , $\frac{\alpha_{1,i}}{\alpha_{2,i}}$

Distinct for different i

Suppose $U \subseteq \mathbb{F}^D$ has a basis $\{v_1, \ldots, v_n\}$ such that each $v_i \in X$.

Problem: Given some other basis $\{u_1, \ldots, u_n\}$ of U, recover $\{v_1, \ldots, v_n\}$ (up to scale).

Example: Jennrich's Algorithm: If $U' = \text{span}\,\{v_1^{\otimes \ell}, \ldots, v_n^{\otimes \ell}\}$ with $\{v_1, \ldots, v_n\}$ linearly independent, then $\{v_1^{\otimes \ell}, \ldots, v_n^{\otimes \ell}\}$ can be recovered from any basis of U' in $D^{O(\ell)}$-time.

Lifted Jennrich's Algorithm [JLV 23, DLCC 07]: Run Jennrich on $U' = U^{\otimes \ell} \cap X^{\ell}$, where $X^{\ell} = \text{span}\{v^{\otimes \ell} : v \in X\}$.

$$v^{\otimes \ell} \in U' \quad \Longleftrightarrow \quad v \in U \cap X$$

Suppose $U \subseteq \mathbb{F}^D$ has a basis $\{v_1, \ldots, v_n\}$ such that each $v_i \in X$.

Problem: Given some other basis $\{u_1, \ldots, u_n\}$ of U, recover $\{v_1, \ldots, v_n\}$ (up to scale).

Example: Jennrich's Algorithm: If $U' = \text{span}\,\{v_1^{\otimes \ell}, \ldots, v_n^{\otimes \ell}\}$ with $\{v_1, \ldots, v_n\}$ linearly independent, then $\{v_1^{\otimes \ell}, \ldots, v_n^{\otimes \ell}\}$ can be recovered from any basis of U' in $D^{O(\ell)}$-time.

Lifted Jennrich's Algorithm [JLV 23, DLCC 07]: Run Jennrich on $U' = U^{\otimes \ell} \cap X^{\ell}$, where $X^{\ell} = \text{span}\{v^{\otimes \ell} : v \in X\}$.

$$v^{\otimes \ell} \in U' \iff v \in U \cap X$$

Theorem (informal) [JLV 23]: Lifted Jennrich's algorithm works already for small ℓ, provided that $R = \dim(U)$ is not too large.

Suppose $U \subseteq \mathbb{F}^D$ has a basis $\{v_1, \dots, v_n\}$ such that each $v_i \in X$.

Problem: Given some other basis $\{u_1, \dots, u_n\}$ of U, recover $\{v_1, \dots, v_n\}$ (up to scale).

Example: Jennrich's Algorithm: If $U' = \operatorname{span}\{v_1^{\otimes \ell}, \dots, v_n^{\otimes \ell}\}$ with $\{v_1, \dots, v_n\}$ linearly independent, then $\{v_1^{\otimes \ell}, \dots, v_n^{\otimes \ell}\}$ can be recovered from any basis of U' in $D^{O(\ell)}$-time.

Lifted Jennrich's Algorithm [JLV 23, DLCC 07]: Run Jennrich on $U' = U^{\otimes \ell} \cap X^\ell$, where $X^\ell = \operatorname{span}\{v^{\otimes \ell} : v \in X\}$.

$$v^{\otimes \ell} \in U' \iff v \in U \cap X$$

Theorem (informal) [JLV 23]: Lifted Jennrich's algorithm works already for small ℓ, provided that $R = \dim(U)$ is not too large.

Example [JLV 23]: If $U \subseteq \mathbb{F}^d \otimes \mathbb{F}^d$ is spanned by $n \leq \frac{1}{4}(d-1)^2$ generic product tensors, then these can be recovered from any basis of U in $\operatorname{poly}(d)$-time.

Corollary [JLV 23]: A generic tensor $T \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^{d^2}$ with

$$\mathrm{rank}(T) \leq \frac{1}{4}(d-1)^2$$

has a unique rank decomposition, that can be recovered in poly(d)-time by applying lifted Jennrich to $\mathrm{im}(T)$.

- Maximum possible rank up to constant
- Quadratic improvement over Jennrich's algorithm, which can handle rank $O(d)$.

Corollary [JLV 23]: A generic tensor $T \in (\mathbb{F}^d)^{\otimes m}$ of tensor rank

$$\mathrm{rank}(T) = O(d^{\lfloor m/2 \rfloor})$$

has a unique tensor rank decomposition, which is recovered in $n^{O(m)}$-time by applying our algorithm to $T\left((\mathbb{F}^d)^{\otimes \lfloor m/2 \rfloor}\right)$.

- r-aided rank: $T = \sum_i v_i \otimes w_i$, where $v_i \in \mathrm{rank} - r$ matrices
- Applications in signal processing and machine learning [Comon, Jutten 2010]

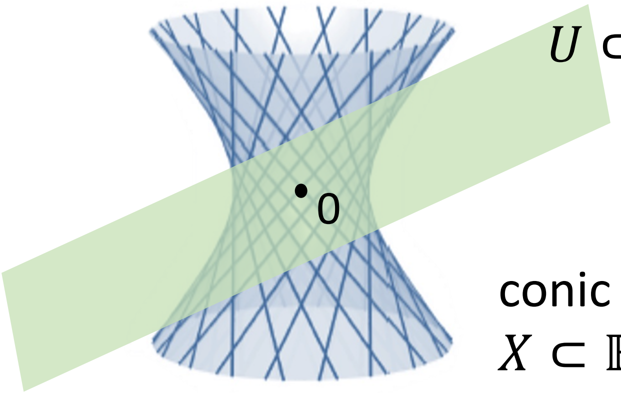
Corollary [JLV 23]: A generic tensor $T \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^k$ of r-aided rank

$$\mathrm{r-aided\ rank}(T) \leq \min\{\Omega_r(d^2), k\}$$

has a unique r-aided rank decomposition, which is recovered in $d^{O(r)}$-time by applying our algorithm to $T\left(\mathbb{F}^k\right)$.

Algorithm: Takeaways

- Natural algorithmic problem
- Captures wide array of decomposition problems
- NP-hard even for X = rank-one matrices

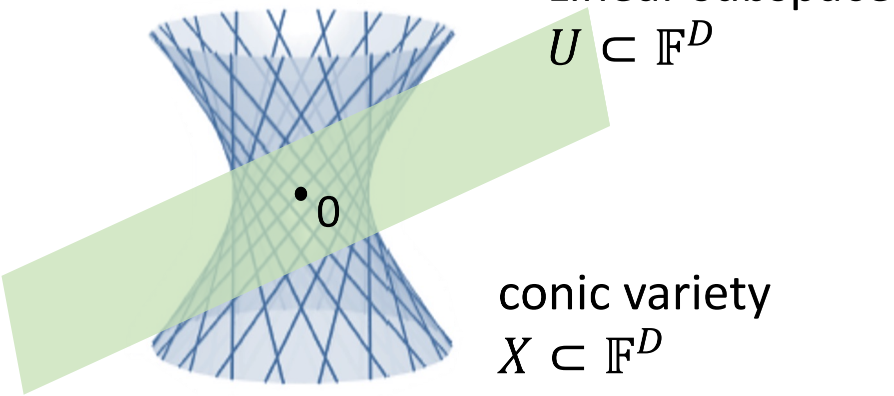


Aim: Compute intersection of variety X and linear subspace U

Main Result: Can design polynomial time algorithm if U is generic and $\dim(U)$ is not too large

Algorithm: Takeaways

- Natural algorithmic problem
- Captures wide array of decomposition problems
- NP-hard even for X = rank-one matrices



Aim: Compute intersection of variety X and linear subspace U

Main Result: Can design polynomial time algorithm if U is generic and $\dim(U)$ is not too large

Future Directions:

- New applications for different choices of varieties?
- Robust versions of the statement?
- Using algebraic geometry ideas for other algorithmic problems

Two goals

$$T = \sum_{a\in[n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

1. <u>Algorithms</u>

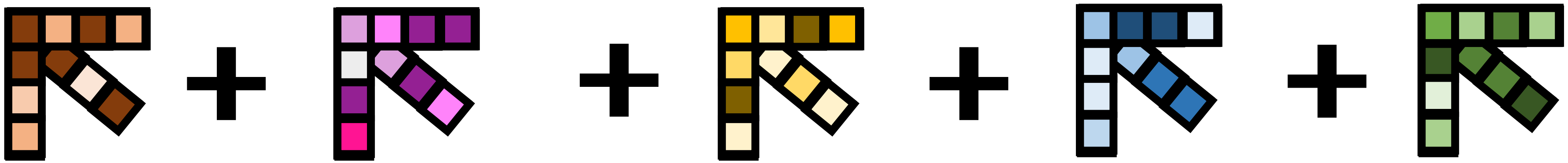
[JLV 2023, published in FOCS]

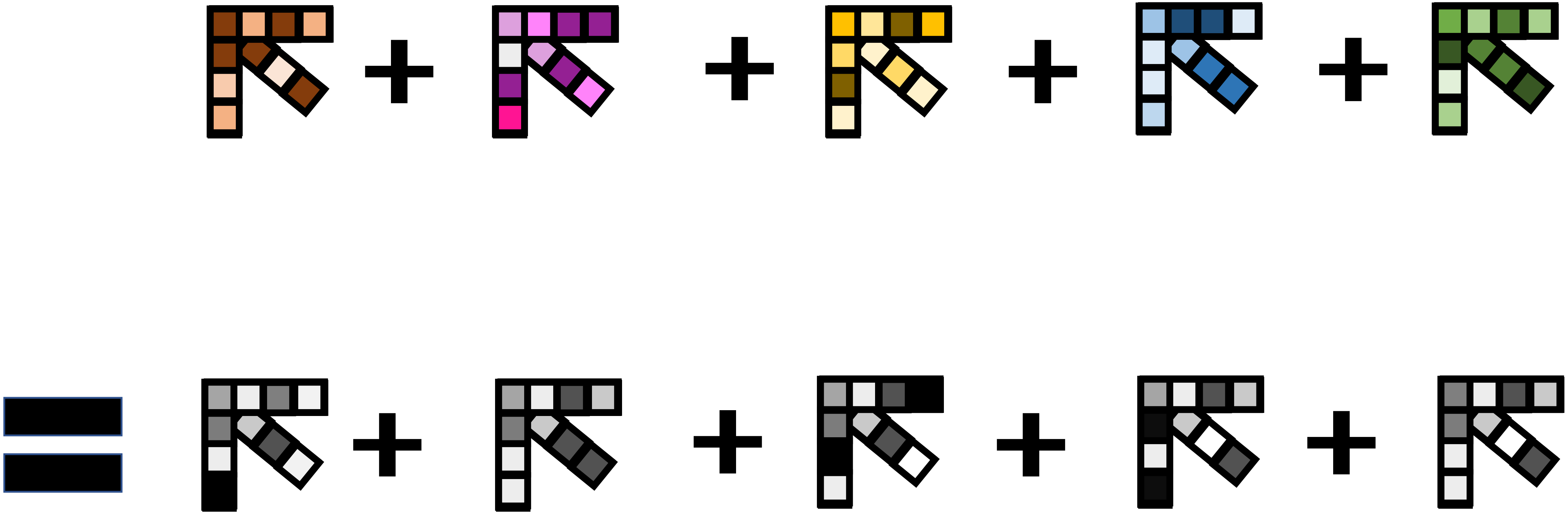
Given a tensor $T \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z}$, find a rank decomposition (1).

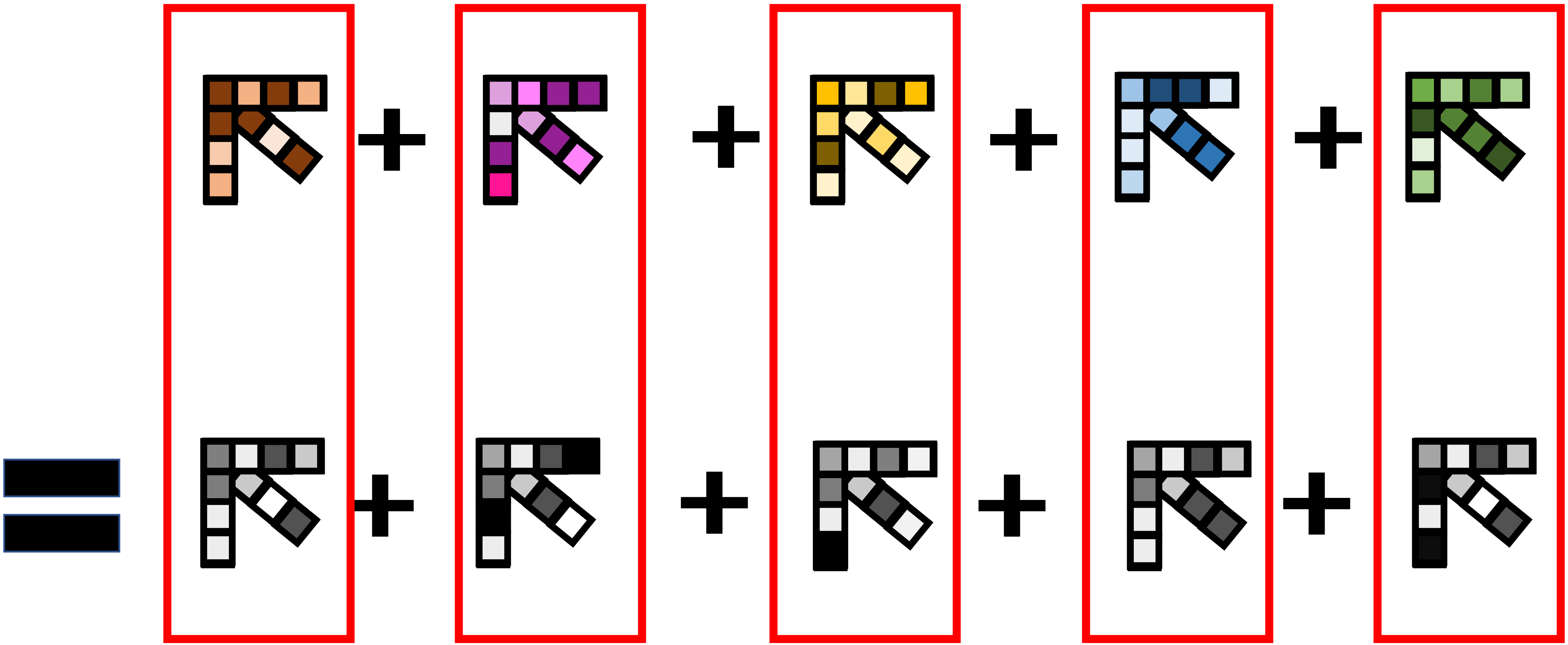
2. <u>Uniqueness</u>

[LP 2023, published in FoM Sigma]

Given a rank decomposition (1), prove that it is the unique rank decomposition.



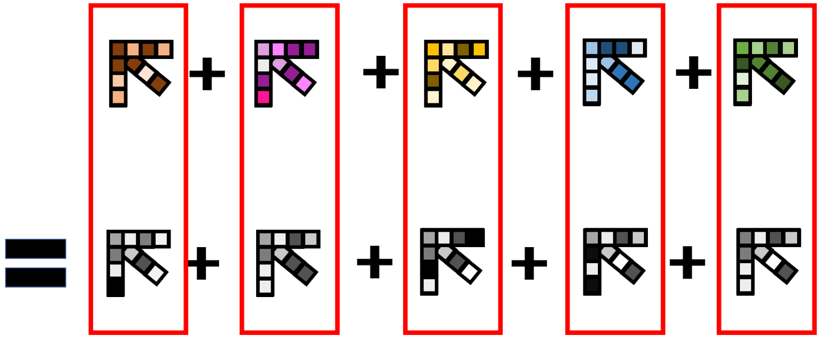




Uniqueness

Jennrich's Uniqueness Theorem: Given a rank decomposition

$$T = \sum_{a \in [d]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^2 \qquad (1)$$



If it holds that

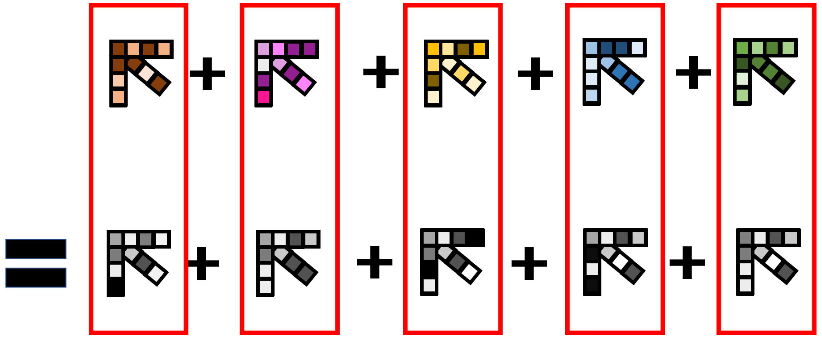
1. $\{x_1, \ldots, x_d\} \subseteq \mathbb{F}^d$ is linearly independent,
2. $\{y_1, \ldots, y_d\} \subseteq \mathbb{F}^d$ is linearly independent,
3. and $\{z_1, \ldots, z_d\} \subseteq \mathbb{F}^2$ are non-parallel

then (1) is the unique rank decomposition of T.

Uniqueness

Jennrich's Uniqueness Theorem: Given a rank decomposition

$$T = \sum_{a \in [d]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^2 \qquad (1)$$



If it holds that

1. $\{x_1, \dots, x_d\} \subseteq \mathbb{F}^d$ is linearly independent,
2. $\{y_1, \dots, y_d\} \subseteq \mathbb{F}^d$ is linearly independent,
3. and $\{z_1, \dots, z_d\} \subseteq \mathbb{F}^2$ are non-parallel

then (1) is the unique rank decomposition of T.

Jennrich's Algorithm: Finds the decomposition (1) efficiently!

Kruskal's theorem

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

Definition: The Kruskal rank of $\{x_1, \ldots, x_n\} \in \mathbb{F}^{d_x}$ is the largest integer k_x such that every subset $S \subseteq \{x_1, \ldots, x_n\}$ of size $|S| = k_x$ is linearly independent.

Kruskal rank is NP-Hard!

Kruskal's theorem

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

Definition: The Kruskal rank of $\{x_1, \ldots, x_n\} \in \mathbb{F}^{d_x}$ is the largest integer k_x such that every subset $S \subseteq \{x_1, \ldots, x_n\}$ of size $|S| = k_x$ is linearly independent.

Kruskal rank is NP-Hard!

$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} + e_4^{\otimes 3} + (e_1 + e_2) \otimes (e_2 + e_3) \otimes (e_1 + e_4)$$

$x_1 \otimes y_1 \otimes z_1$ ... x_5 $\otimes$ y_5 $\otimes$ z_5

$$\{x_1, \ldots, x_5\} = \{e_1, e_2, e_3, e_4, e_1 + e_2\}, \qquad k_x = 2.$$

Kruskal's theorem

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

Definition: The Kruskal rank of $\{x_1, \dots, x_n\} \in \mathbb{F}^{d_x}$ is the largest integer k_x such that every subset $S \subseteq \{x_1, \dots, x_n\}$ of size $|S| = k_x$ is linearly independent.

Kruskal rank is NP-Hard!

Kruskal's theorem: If $2n \leq k_x + k_y + k_z - 2$, then (1) is the unique rank decomposition of T.

Example [Jennrich's Theorem]: $k_x = k_y = n$ and $k_z \geq 2$.

Kruskal's theorem

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

Definition: The Kruskal rank of $\{x_1, \ldots, x_n\} \in \mathbb{F}^{d_x}$ is the largest integer k_x such that every subset $S \subseteq \{x_1, \ldots, x_n\}$ of size $|S| = k_x$ is linearly independent.

Kruskal rank is NP-Hard!

Kruskal's theorem: If $2n \leq k_x + k_y + k_z - 2$, then (1) is the unique rank decomposition of T.

Example [Jennrich's Theorem]: $k_x = k_y = n$ and $k_z \geq 2$.

$\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are linearly independent

Matroid theory for product tensors

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

- Recall the general setup: We are handed a set of product tensors $\{x_a \otimes y_a \otimes z_a : a \in [n]\}$, and want to determine if their sum (1) is a unique rank decomposition.

Matroid theory for product tensors

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

- Recall the general setup: We are handed a set of product tensors $\{x_a \otimes y_a \otimes z_a : a \in [n]\}$, and want to determine if their sum (1) is a unique rank decomposition.

- Natural tool: Matroid theory (the study of finite sets of vectors).

Matroid theory for product tensors

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

- Recall the general setup: We are handed a set of product tensors $\{x_a \otimes y_a \otimes z_a : a \in [n]\}$, and want to determine if their sum (1) is a unique rank decomposition.

- Natural tool: Matroid theory (the study of finite sets of vectors).

- Line of attack: Determine matroidal properties of sets of product tensors.

Matroid theory for product tensors

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

- Recall the general setup: We are handed a set of product tensors $\{x_a \otimes y_a \otimes z_a : a \in [n]\}$, and want to determine if their sum (1) is a unique rank decomposition.

- Natural tool: Matroid theory (the study of finite sets of vectors).

- Line of attack: Determine matroidal properties of sets of product tensors.

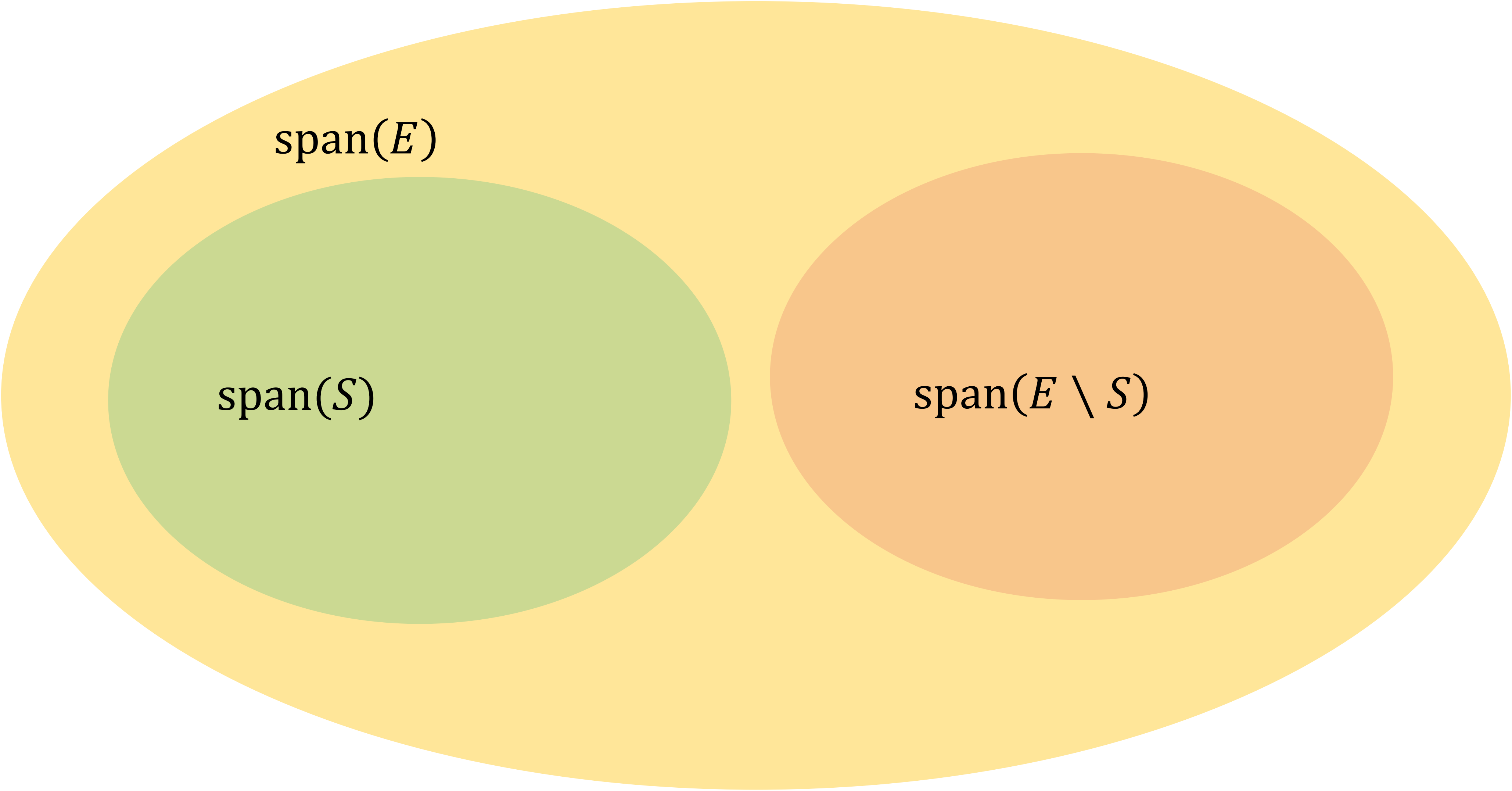
Rest of talk: A splitting theorem for product tensors

Splitting

Definition: A set of vectors $E = \{v_1, \ldots, v_n\}$ splits if there exists a non-trivial subset $S \subseteq E$ such that

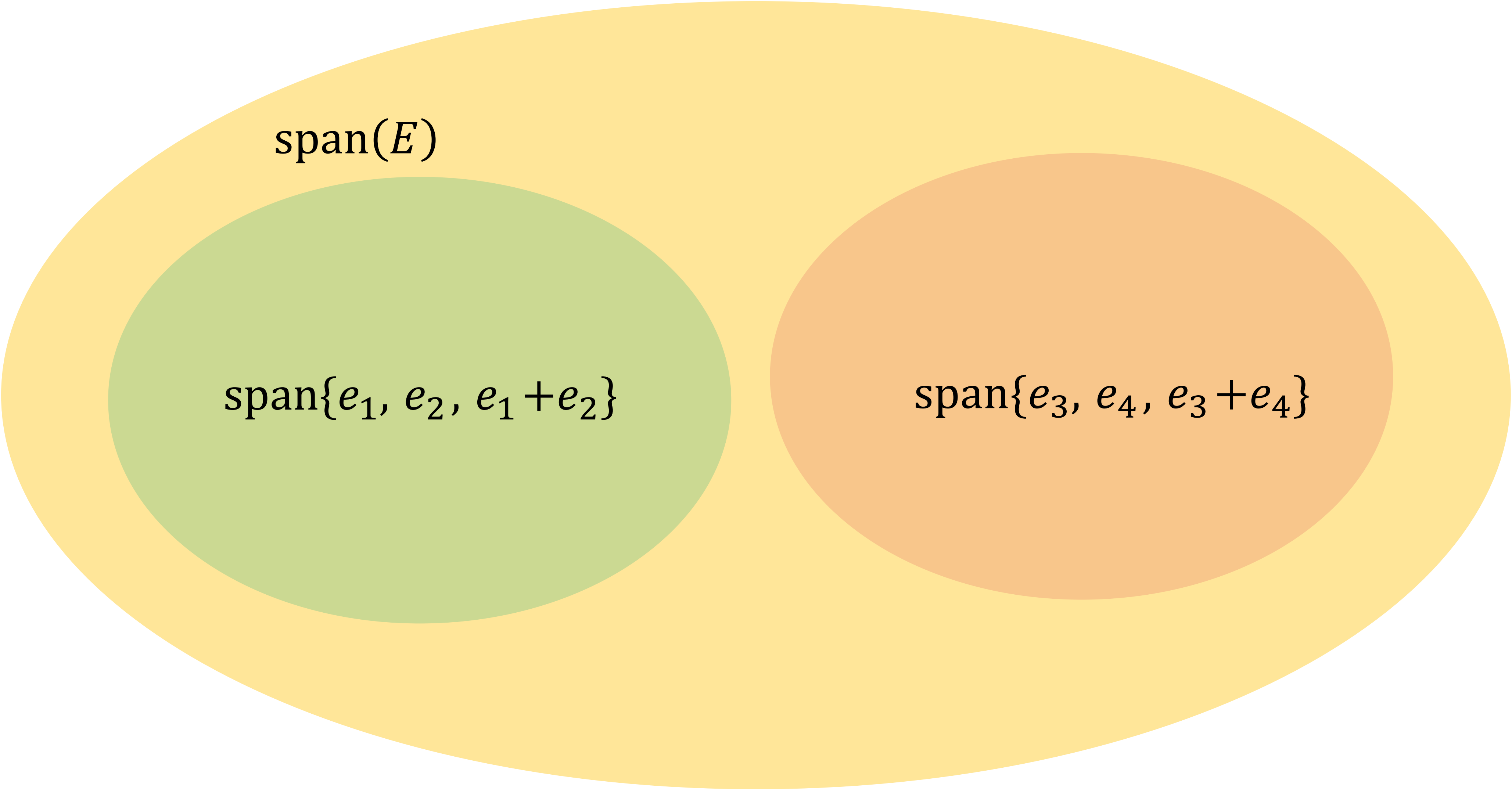
$$\operatorname{span}(S) \cap \operatorname{span}(E \setminus S) = \{0\} \tag{2}$$

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$



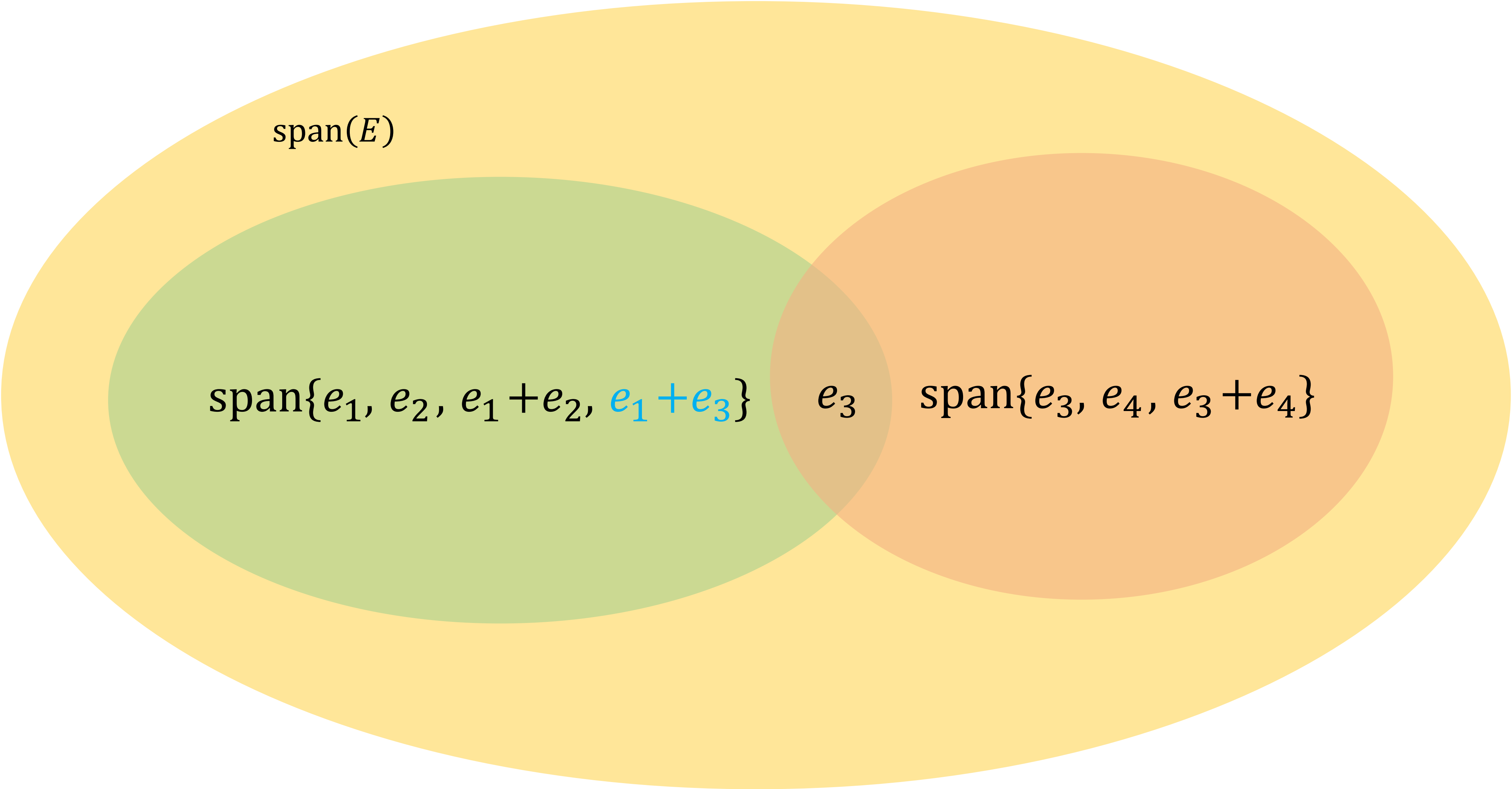
E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

$E = \{e_1, e_2, e_1+e_2, e_3, e_4, e_3+e_4\}$



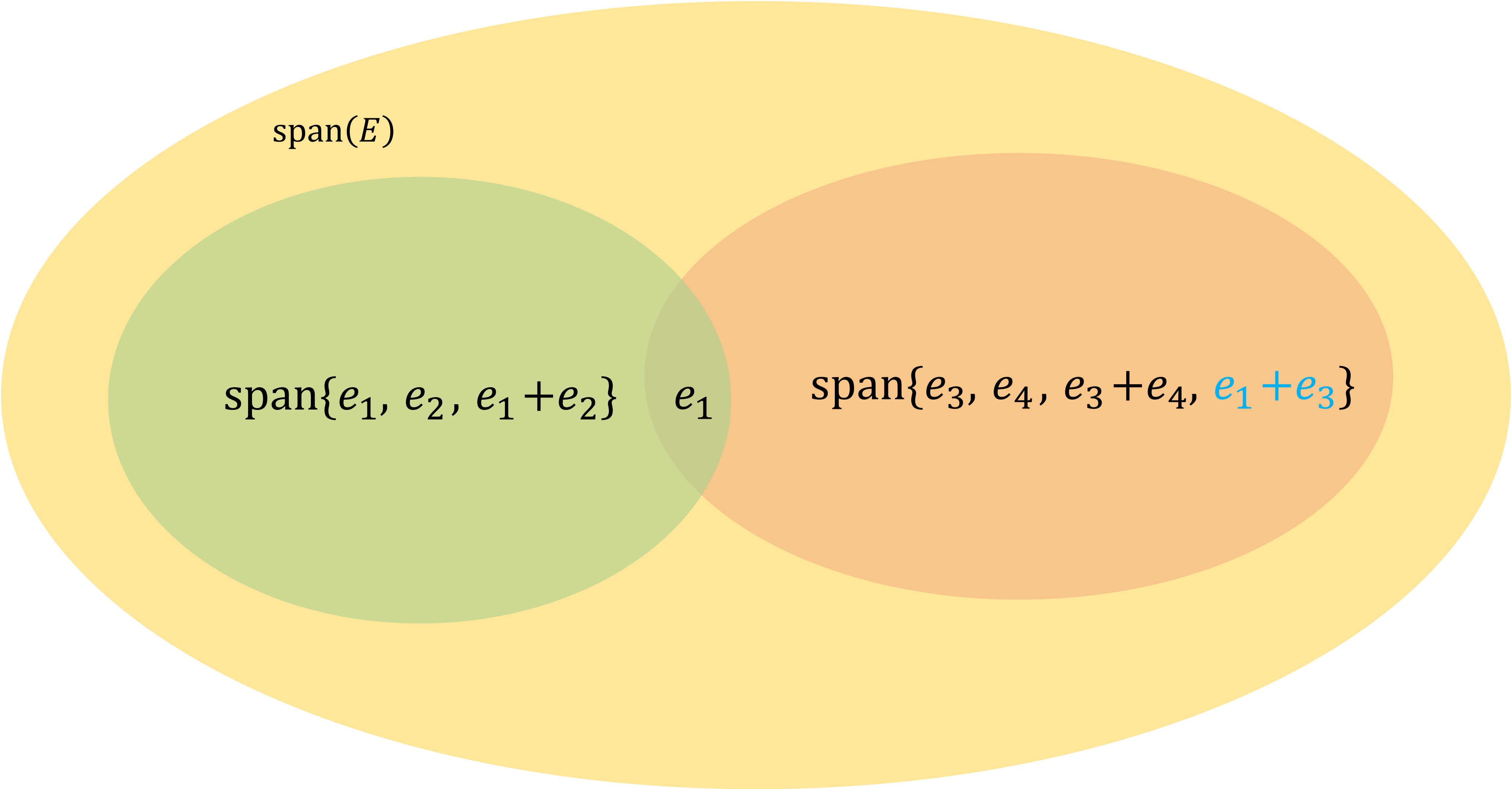
E splits if there exists $S \subseteq E$ such that $\text{span}(S) \cap \text{span}(E \setminus S) = \{0\}$

$$E = \{e_1, e_2, e_1+e_2, e_3, e_4, e_3+e_4\} \cup \{e_1+e_3\}$$



E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

$E = \{e_1, e_2, e_1+e_2, e_3, e_4, e_3+e_4\} \cup \{e_1+e_3\}$



Splitting

<u>Definition:</u> A set of vectors $E = \{v_1, \ldots, v_n\}$ splits if there exists a non-trivial subset $S \subseteq E$ such that

$$\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\} \qquad (2)$$

Splitting

Definition: A set of vectors $E = \{v_1, \dots, v_n\}$ splits if there exists a non-trivial subset $S \subseteq E$ such that

$$\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\} \qquad (2)$$

Fact: If (2) holds, and $\sum(E) = 0$, then $\sum(S) = \sum(E \setminus S) = 0$

Splitting

Definition: A set of vectors $E = \{v_1, \dots, v_n\}$ splits if there exists a non-trivial subset $S \subseteq E$ such that

$$\text{span}(S) \cap \text{span}(E \setminus S) = \{0\} \qquad (2)$$

Fact: If (2) holds, and $\sum(E) = 0$, then $\sum(S) = \sum(E \setminus S) = 0$

Proof: $\sum(E) = 0$

$$\Rightarrow \sum(S) = -\sum(E \setminus S) \in \text{span}(S) \cap \text{span}(E \setminus S) = \{0\}$$

∎

Splitting theorem

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

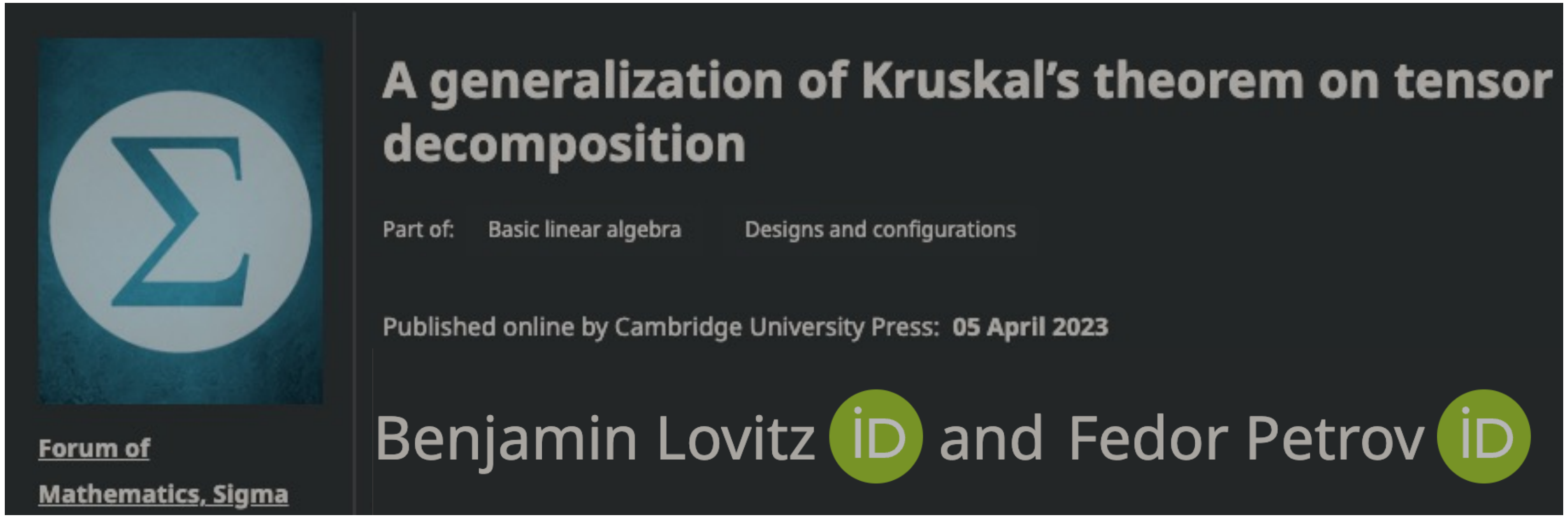
Splitting theorem [LP 2023]: Let $E = \{x_a \otimes y_a : a \in [n]\}$.

If

$$\dim\mathrm{span}(E) \leq d_x^{[n]} + d_y^{[n]} - 2$$

then E splits.

$d_x^{[n]} = \dim\mathrm{span}\{x_1, \dots, x_n\}$



Splitting theorem

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

Splitting theorem [LP 2023]: Let $E = \{x_a \otimes y_a : a \in [n]\}$.

If

$$\mathrm{dimspan}(E) \leq d_x^{[n]} + d_y^{[n]} - 2$$

then E splits.

$d_x^{[n]} = \mathrm{dimspan}\{x_1, \ldots, x_n\}$

Corollary: A uniqueness result that is stronger than Jennrich's (and Kruskal's!)

Splitting theorem

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

Splitting theorem [LP 2023]: Let $E = \{x_a \otimes y_a : a \in [n]\}$.

If

$$\mathrm{dimspan}(E) \leq d_x^{[n]} + d_y^{[n]} - 2$$

then E splits.

$d_x^{[n]} = \mathrm{dimspan}\{x_1, \dots, x_n\}$

Corollary: A uniqueness result that is stronger than Jennrich's (and Kruskal's!)

More matroid theory for product tensors?

Splitting theorem

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

<u>Splitting theorem [LP 2023]:</u> Let $E = \{x_a \otimes y_a \otimes z_a : a \in [n]\}$.

If

$$\mathrm{dimspan}(E) \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 3$$

then E splits.

$d_x^{[n]} = \mathrm{dimspan}\{x_1, \ldots, x_n\}$

Splitting theorem

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

Splitting theorem [LP 2023]: Let $E = \{x_a \otimes y_a \otimes z_a : a \in [n]\}$.

If

$$\mathrm{dimspan}(E) \le d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 3$$

$$d_x^{[n]} = \mathrm{dimspan}\{x_1, \dots, x_n\}$$

then E splits.

Corollary: If

$$n \le d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then E splits.

Splitting theorem

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

<u>Splitting theorem [LP 2023]:</u> Let $E = \{x_a \otimes y_a \otimes z_a : a \in [n]\}$.

If

$$\dim\mathrm{span}(E) \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 3$$

$$d_x^{[n]} = \dim\mathrm{span}\{x_1, \dots, x_n\}$$

then E splits.

<u>Corollary:</u> If

$$n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then E splits.

Splitting theorem => Corollary: If E is linearly independent, then it splits. Otherwise,

$$\dim\mathrm{span}(E) \leq n - 1 \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 3,$$

so E splits by splitting theorem. ∎

Splitting theorem

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

Splitting theorem [LP 2023]: Let $E = \{x_a \otimes y_a \otimes z_a : a \in [n]\}$.

If

$$\dim\mathrm{span}(E) \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 3$$

then E splits.

$$d_x^{[n]} = \dim\mathrm{span}\{x_1, \dots, x_n\}$$

Corollary: If

$$n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then E splits.

Replaces Kruskal ranks with standard ranks

Corollary: If $2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2$, then for any other set of product tensors $E' = \{x'_a \otimes y'_a \otimes z'_a : a \in [n]\}$, $E \cup E'$ splits.

Recall the tensor decomposition setup…

We are handed a rank decomposition

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a$$

… and want to control other rank decompositions

$$T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a.$$

Recall the tensor decomposition setup…

We are handed a rank decomposition

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a$$

… and want to control other rank decompositions

$$T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a.$$

Replaces Kruskal ranks with standard ranks

<u>Corollary:</u> If $2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2$, then for any other set of product tensors $E' = \{x'_a \otimes y'_a \otimes z'_a : a \in [n]\}$, $E \cup E'$ splits.

Corollary => Kruskal generalization

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

Corollary [LP 2023]: If

$d_x^{[n]} = \text{dimspan}\{x_1, \dots, x_n\}$

$$2n \le d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

Poly time to check!

then for any other decomposition $T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a$

there exist non-trivial subsets $S, T \subseteq [n]$ such that $\sum_{a \in S} x_a \otimes y_a \otimes z_a = \sum_{a \in T} x'_a \otimes y'_a \otimes z'_a$

Corollary => Kruskal generalization

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

<u>Corollary [LP 2023]:</u> If

$d_x^{[n]} = \mathrm{dimspan}\{x_1, \ldots, x_n\}$

$$2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

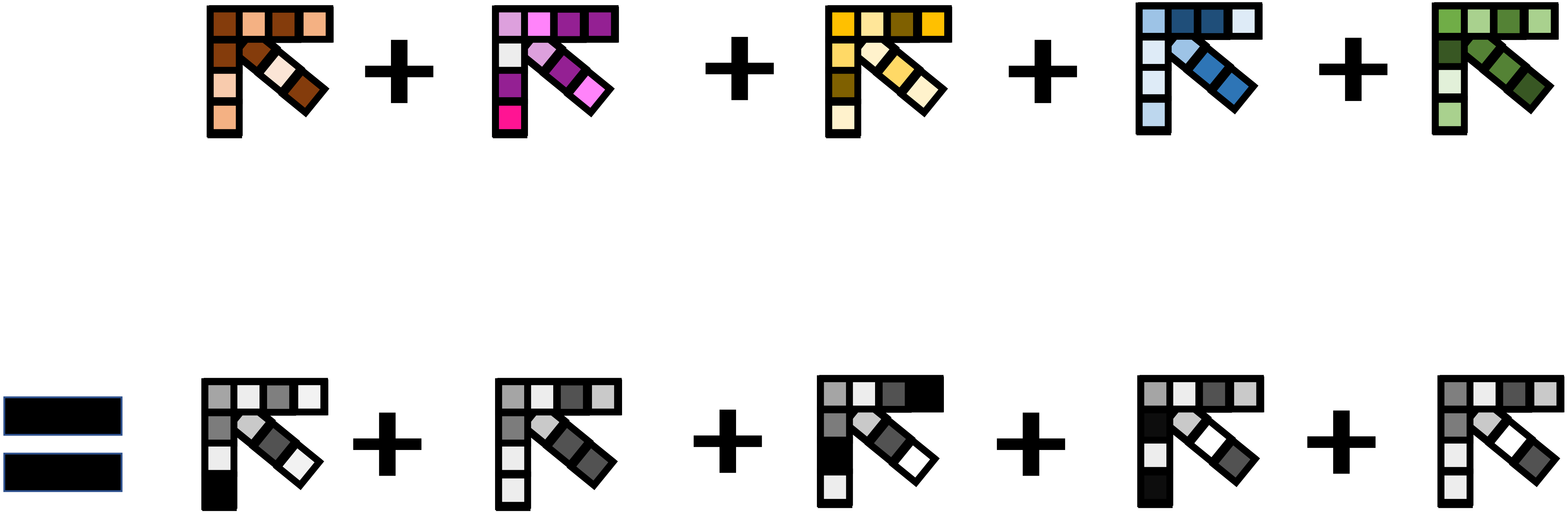
Poly time to check!

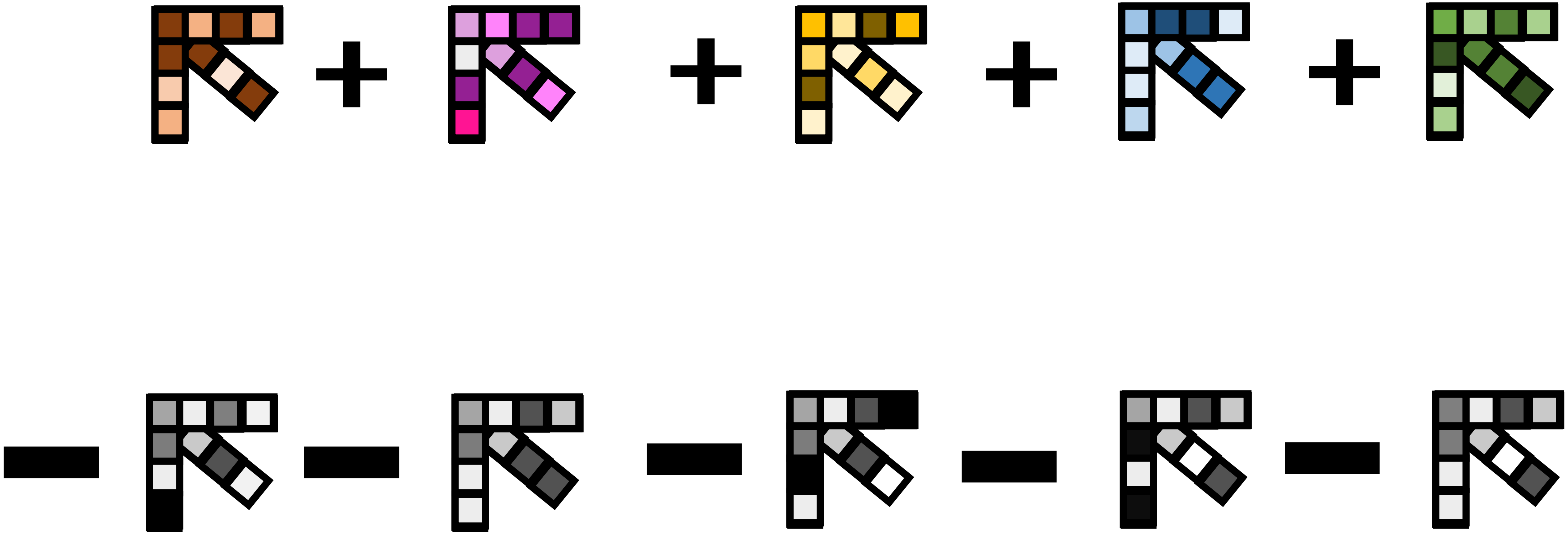
then for any other decomposition $T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a$

there exist non-trivial subsets $S, T \subseteq [n]$ such that $\sum_{a \in S} x_a \otimes y_a \otimes z_a = \sum_{a \in T} x'_a \otimes y'_a \otimes z'_a$

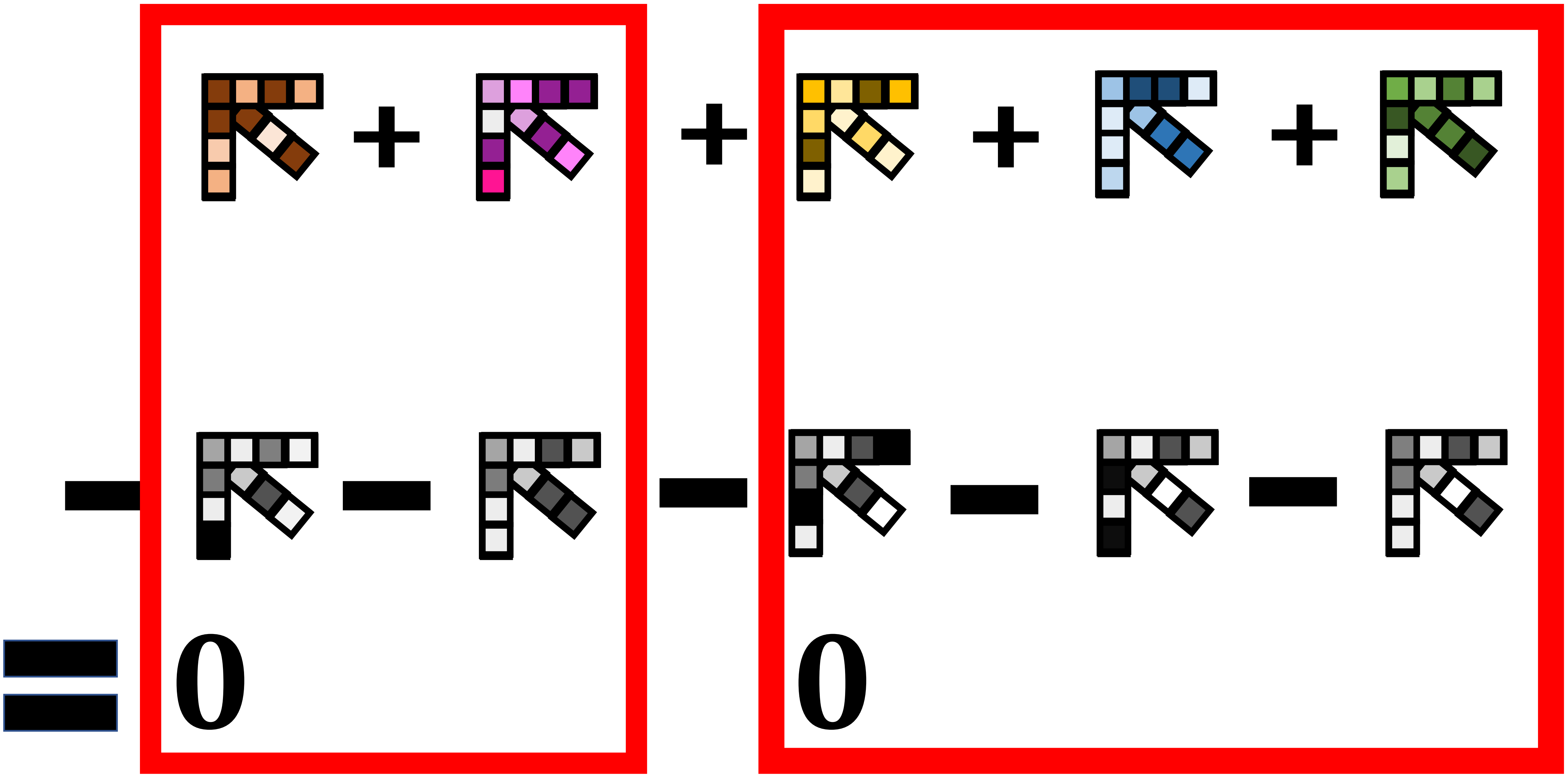
<u>Proof:</u>

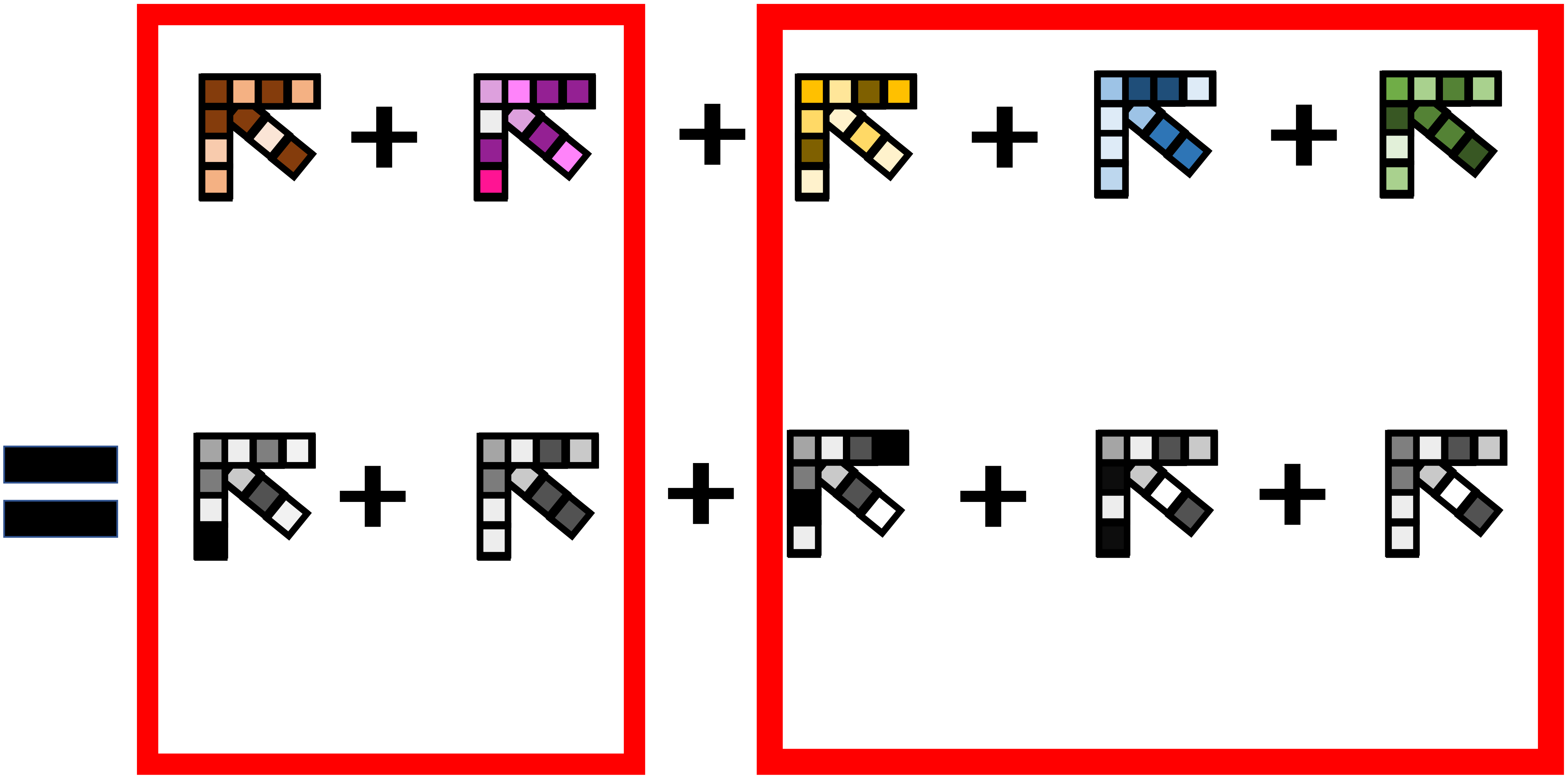
By previous corollary, $\{x_a \otimes y_a \otimes z_a, x'_a \otimes y'_a \otimes z'_a \; : a \in [n]\}$ splits





= 0


0
0



$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \qquad (1)$$

Corollary [L-Petrov]: If

$d_x^{[n]} = \operatorname{dimspan}\{x_1, \ldots, x_n\}$

$$2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

Poly time to check!

then for any other decomposition $T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a$

there exist non-trivial subsets $S, R \subseteq [n]$ such that

$$\sum_{a \in S} x_a \otimes y_a \otimes z_a = \sum_{a \in R} x'_a \otimes y'_a \otimes z'_a$$

Uniqueness

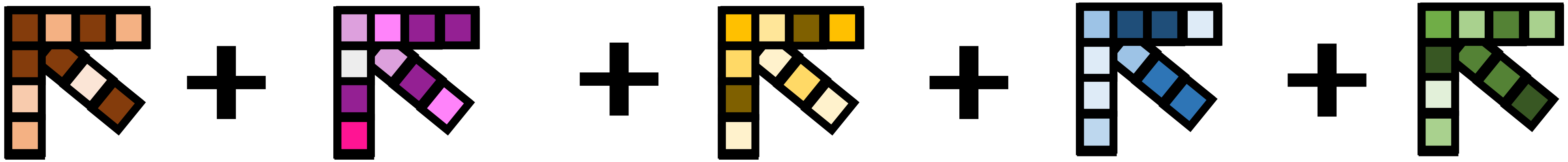
$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \quad (1)$$

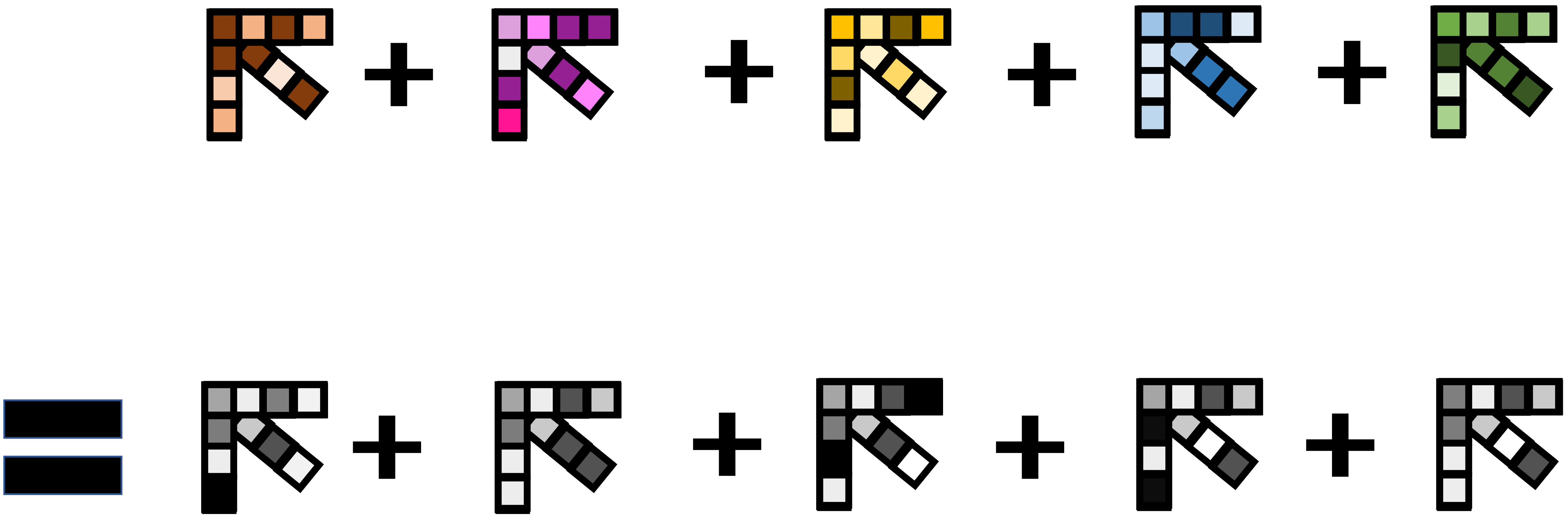
Theorem [LP 2023]: If for every subset $S \subseteq [n]$ of size $|S| \geq 2$, it holds that

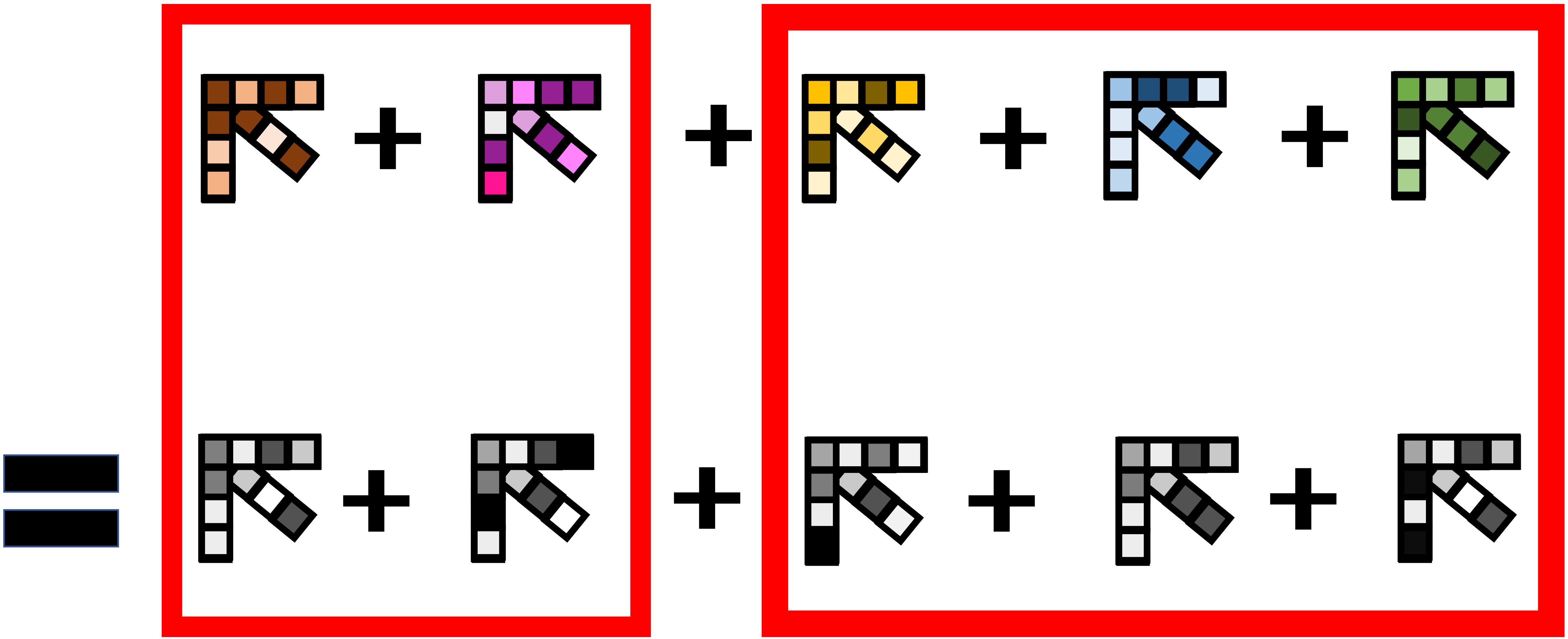
$$2|S| \leq d_x^S + d_y^S + d_z^S - 2,$$

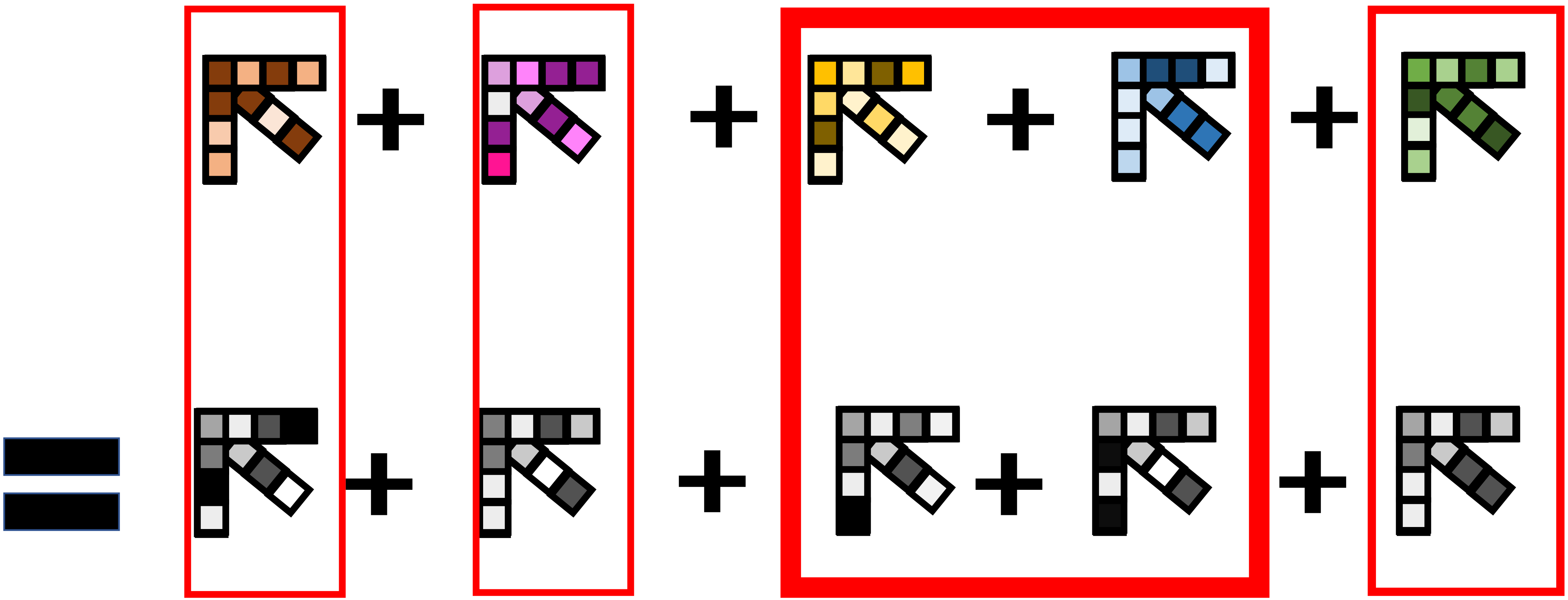
$d_x^S = \mathrm{dimspan}\{x_a : a \in S\}$

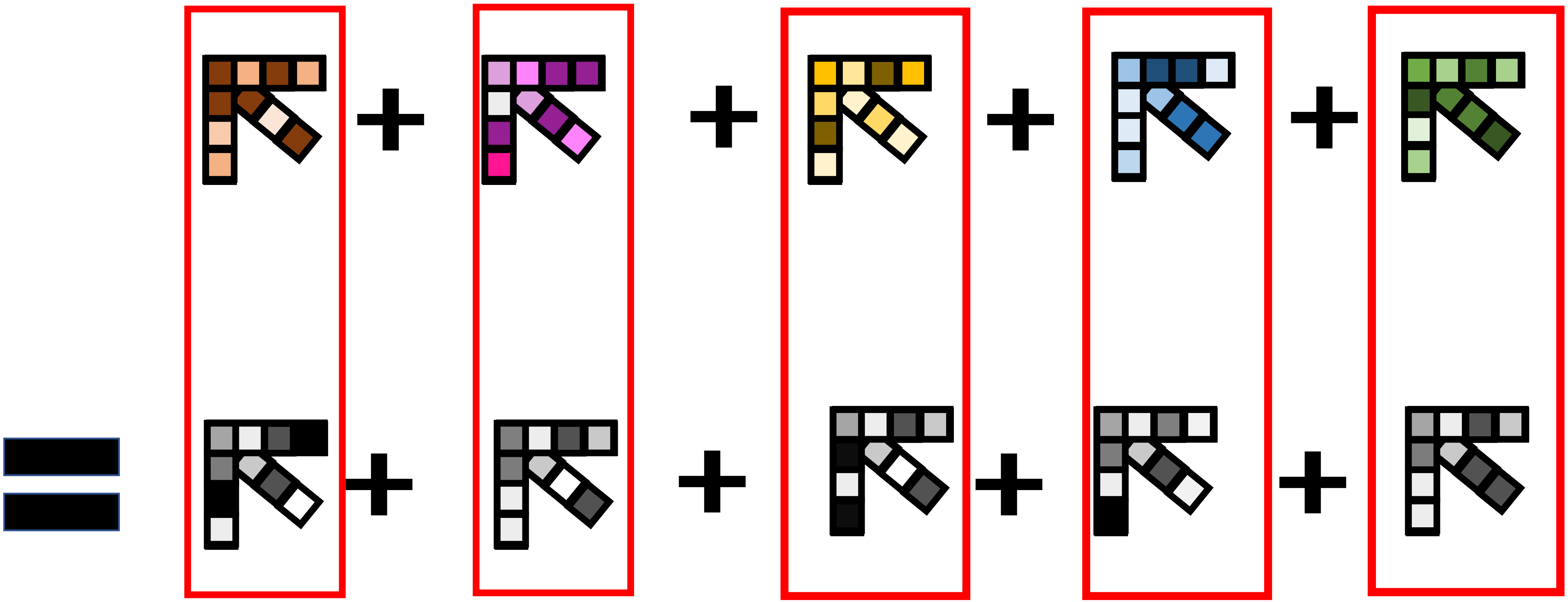
then (1) is the unique rank decomposition of T.











Uniqueness

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in \mathbb{F}^{d_x} \otimes \mathbb{F}^{d_y} \otimes \mathbb{F}^{d_z} \quad (1)$$

Theorem [LP 2023]: If for every subset $S \subseteq [n]$ of size $|S| \geq 2$, it holds that

$$2|S| \leq d_x^S + d_y^S + d_z^S - 2,$$

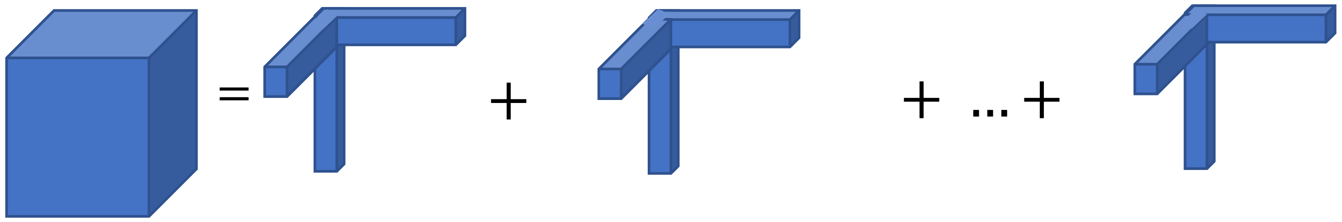
$d_x^S = \mathrm{dimspan}\{x_a : a \in S\}$

then (1) is the unique rank decomposition of T.

Conclusion

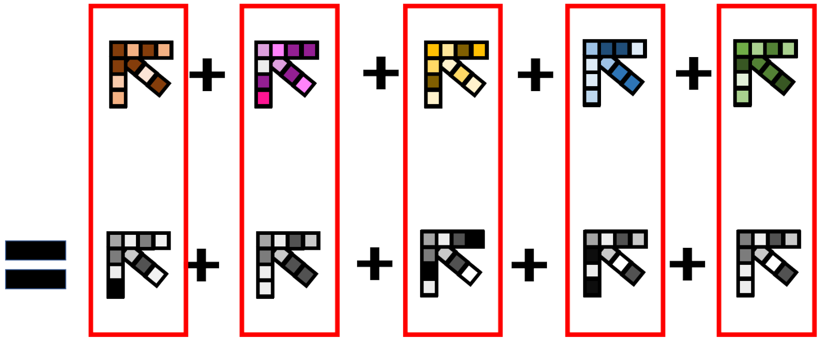
__Algorithms:__

- Intersecting variety X with subspace $\leftrightarrow$ (X, k)-decompositions
- Broad applications for different choices of X
- In particular, can decompose tensors of quadratically higher rank than Jennrich



__Uniqueness:__

- Splitting theorem "demystifies" Kruskal's theorem
- More matroid theory for product tensors?



Algorithms and Uniqueness of Tensor Decompositions

Benjamin Lovitz

Northeastern University

UMass Boston Department of Mathematics

November 14, 2023