New techniques for bounding stabilizer rank

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Stabilizer states

- A state $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ a stabilizer state if $|\phi\rangle = U|0\rangle$ for some Clifford circuit U.

Recall: A Clifford circuit is a circuit composed of Clifford gates

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Stabilizer states

- A state $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ a stabilizer state if $|\phi\rangle = U|0\rangle$ for some Clifford circuit U.

- Theorem [Dehaene, De Moor 03]:

$$|\phi\rangle \text{ stabilizer} \Leftrightarrow |\phi\rangle = \sum_{x \in A} i^{l(x)} (-1)^{q(x)} |x\rangle,$$

where $A \subseteq \mathbb{F}_2^n$ is an affine linear subspace

$l: \mathbb{F}_2^n \to \mathbb{F}_2$ is a linear function

$q: \mathbb{F}_2^n \to \mathbb{F}_2$ is a quadratic polynomial

Key takeaway: $|\phi\rangle$ stabilizer $\Rightarrow$coordinates are in $\{0, \pm 1, \pm i\}$

Stabilizer rank

- The stabilizer rank of a state

$$|\psi\rangle \in (\mathbb{C}^2)^{\otimes n},$$

denoted $\chi(|\psi\rangle)$, is the minimum number r for which

$$|\psi\rangle = \sum_{i=1}^{r} c_i |\phi_i\rangle$$

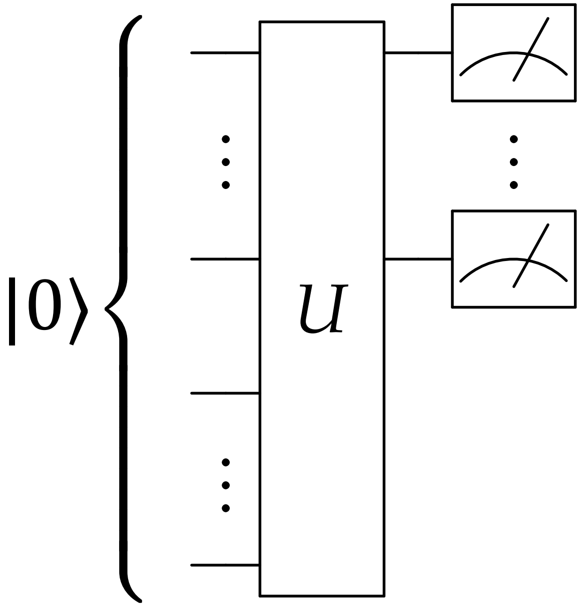
for some $c_i \in \mathbb{C}$ and $|\phi_i\rangle$ stabilizer states.

- The δ-approximate stabilizer rank of $|\psi\rangle$ is

$$\chi_\delta(|\psi\rangle) = \min\{\chi(|\mu\rangle) : \| |\psi\rangle - |\mu\rangle \| \leq \delta\}.$$

Motivation: Classical simulation

Question: Given a quantum circuit



Can it be simulated efficiently on a classical computer?

Motivation: Classical simulation

Answer: Write U as a sequence of Clifford+T gates (universal gate set).

Clifford gates: $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$, $CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

T gate: $\mathrm{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

[Gottesman-Knill 98]: U Clifford $\Rightarrow$ Yes.

$|H\rangle = |0\rangle + (1+\sqrt{2})|1\rangle \approx |0\rangle + 2.41|1\rangle$

[Bravyi-Smith-Smolin 16, Bravyi-Gosset 16]: If U consists of n T gates, then circuit can be simulated with cost quadratic in $\chi(|H\rangle^{\otimes n})$.

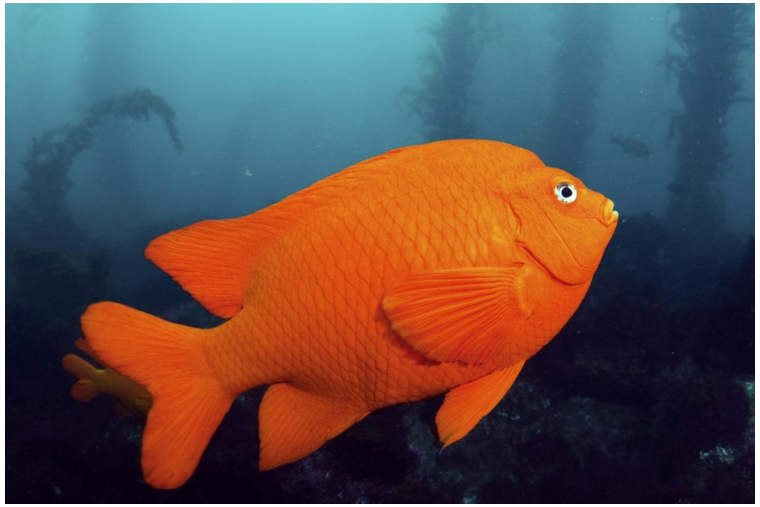
… and can be approximately simulated with cost quadratic in $\chi_\delta(|H\rangle^{\otimes n})$.

This talk

Lower bounds

Match [Peleg, Shpilka, Volk 22] up to log factor

- $\chi\left(|H\rangle^{\otimes n}\right) \geq \Omega(n/\log n)$.
- There exists $\delta > 0$ such that $\chi_\delta\left(|H\rangle^{\otimes n}\right) \geq \Omega(\sqrt{n}/\log n)$.



Upper bounds

- Generic stabilizer rank

Open questions

- Better bounds?

Image: https://fishingbooker.com/blog/california-state-fish-golden-trout-garibaldi/

Subset-sum representations

- Let $\alpha \in \mathbb{C}^k, \beta \in \mathbb{C}^r$. We say β is a subset-sum representation of α if each α_j is equal to the sum of some subset of $\{\beta_1, \dots, \beta_r\}$.

- Example: $\beta = (1,2)$ is a subset-sum representation of $\alpha = (0,1,2,3)$.

- Example: If $|\psi\rangle = \sum_{i=1}^{r} c_i |\phi_i\rangle$ is a stabilizer decomposition, then
$$\beta = (c_1, \dots, c_r, -c_1, \dots, -c_r, ic_1, \dots, ic_r, -ic_1, \dots, -ic_r) \in \mathbb{C}^{4r}$$
is a subset-sum representation of $|\psi\rangle$.

$|\phi_i\rangle$ stabilizer $\Rightarrow$coordinates are in $\{0, \pm 1, \pm i\}$

$\Rightarrow \chi(|\psi\rangle) \geq \frac{1}{4} \cdot$ (the size of the smallest subset-sum rep of $|\psi\rangle$)

Lower bounds on the size of a subset-sum rep

- Let $\alpha \in \mathbb{C}^k, \beta \in \mathbb{C}^r$. We say β is a subset-sum representation of α if each α_j is equal to the sum of some subset of $\{\beta_1, \dots, \beta_r\}$.

- Trivially, $\mathrm{r} \geq \log_2 k$, since $\{\beta_1, \dots, \beta_r\}$ has just 2^r different subsets.

α exponentially increasing

- Theorem [Moulton 01]: If $2|\alpha_j| \leq |\alpha_{j+1}|$ for all $j \in \{1, \dots, k-1\}$, then $r \geq k/\log_2 k$.

Linear in k, instead of logarithmic!

- Example: If $\alpha = (1,2,4, \dots, 2^{k-1})$, then $r \geq k/\log_2 k$

- Theorem [Lovitz-Steffan]: If the coordinates of $|\psi\rangle$ contain an exponentially increasing sequence of length k, then $\chi(|\psi\rangle) \geq k/(4\log_2 k)$.

Lower bound on stabilizer rank

- Theorem [Lovitz-Steffan]: If the coordinates of $|\psi\rangle$ contain an exponentially increasing sequence of length k, then $\chi(|\psi\rangle) \geq k/(4\log_2 k)$.

Corollary [Lovitz-Steffan]: $\chi(|H\rangle^{\otimes n}) \geq n/(4\log_2 n)$.

Proof: Since $|H\rangle \approx |0\rangle + 2.41|1\rangle$,

$$|H\rangle^{\otimes n} \approx |0\cdots 0\rangle + (2.41)(|0\cdots 01\rangle + \cdots + |10\cdots 0\rangle) + \cdots + (2.41)^n|1\cdots 1\rangle.$$

$\Rightarrow |H\rangle^{\otimes n}$ contains the exponentially increasing sequence $(2.41, 2.41^2, \ldots, 2.41^n)$

$\Rightarrow \chi(|H\rangle^{\otimes n}) \geq n/(4\log_2 n)$ by boxed theorem.

Lower bound on approximate stabilizer rank

- The δ-approximate stabilizer rank of a normalized state $|\psi\rangle$ is
$$\chi_\delta(|\psi\rangle) = \min\{\chi(|\mu\rangle) : \||\psi\rangle - |\mu\rangle\| \leq \delta\}.$$

- <u>Theorem [Lovitz-Steffan]:</u> There exists $\delta > 0$ for which $\chi_\delta(|H\rangle^{\otimes n}) \geq \sqrt{n}/(4 \log_2 \sqrt{n})$.

Proof sketch: Since the coordinates of $|H\rangle^{\otimes n}$ contain an exponentially increasing sequence of length n, then for δ small enough, the coordinates of $|\mu\rangle$ contain an exponentially increasing sequence of length $\sqrt{n}$ (De Moivre-Laplace).

Result follows from boxed theorem. ∎

This talk

Lower bounds

Match [Peleg, Shpilka, Volk 22] up to log factor

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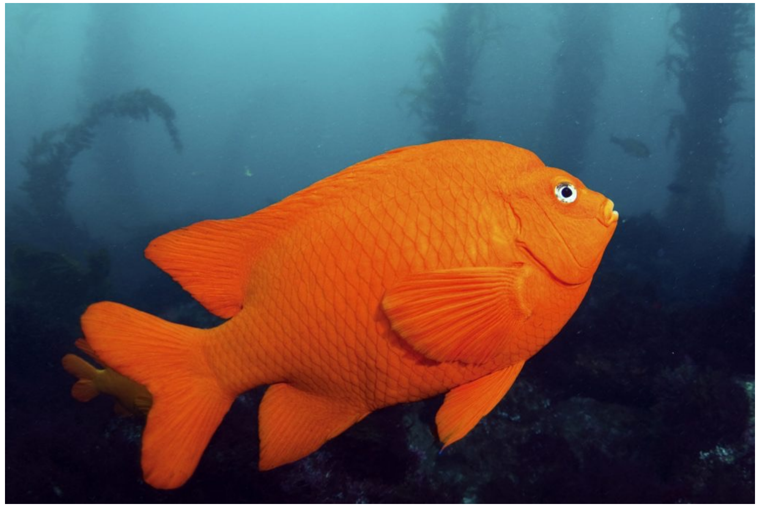
- There exists $\delta > 0$ such that $\chi_\delta\left(|H\rangle^{\otimes n}\right) \geq \Omega(\sqrt{n}/\log n)$.



Upper bounds

- Generic stabilizer rank

Open questions

- Better bounds?

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Upper bounds: Generic stabilizer rank

- Let $\chi_n = \max\{\chi(|\psi\rangle^{\otimes n}) \colon |\psi\rangle \in \mathbb{C}^2\}$ be the n-th generic stabilizer rank.
- $\chi_n \geq \chi(|H\rangle^{\otimes n})$
- Fact: $\chi(|\psi\rangle^{\otimes n}) = \chi_n$ for all but finitely many $|\psi\rangle \in \mathbb{C}^2$ (up to scale).
- Proposition [Lovitz-Steffan]: $\chi_n = O(2^{n/2})$

(Slight improvement of recent bound $O((n+1)2^{n/2})$ of [Qassim-Pashayan-Gosset 21])

- Proposition [Lovitz-Steffan]: There exists a single set of χ_n stabilizer states that can be superimposed to produce any state of the form $|\psi\rangle^{\otimes n}$.

This talk

<u>Lower bounds</u>

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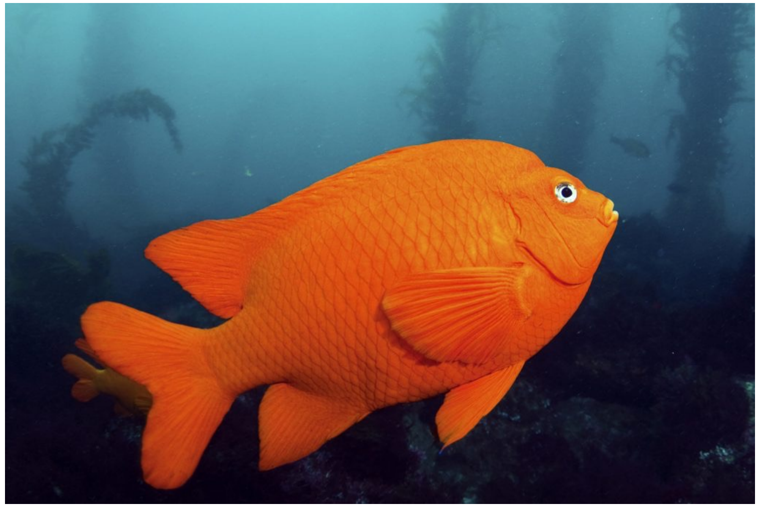
Match [Peleg, Shpilka, Volk 22] up to log factor



<u>Upper bounds</u>

- Generic stabilizer rank

<u>Open questions</u>

- Better bounds?

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Open questions

- Better lower bounds on $\chi\left(|H\rangle^{\otimes n}\right)$?

[BSS16] idea: Find states $|\psi_n\rangle \in (\mathbb{C}^2)^{\otimes n}$ of low T-count and high stabilizer rank

For example, if there exist states $|\psi_n\rangle$ that:

1. Have T-count $O(2^{cn})$ for some $0 < c < 1$.
2. Contain an exponentially increasing sequence of coordinates of length $\Omega(2^n)$.

... then $\chi\left(|H\rangle^{\otimes n}\right) = \Omega\left(n^{1/c} / \log_2(n)\right)$.

Super-linear

- Better upper bounds on χ_n?