A generalization of Kruskal’s theorem

What happens when you replace the Kruskal ranks with standard ranks in Kruskal’s theorem?

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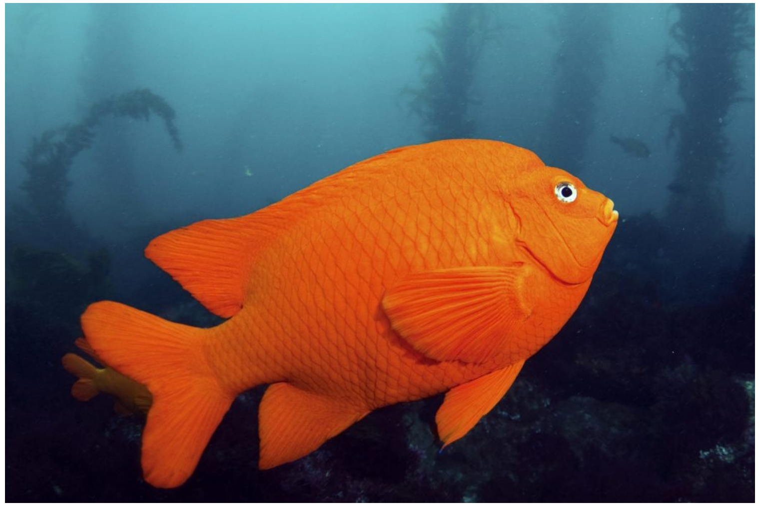
Slides available at www.benjaminlovitz.com

Outline

- Uniqueness of tensor decompositions
- Kruskal's theorem
- A generalization of Kruskal's theorem
- Splitting theorem



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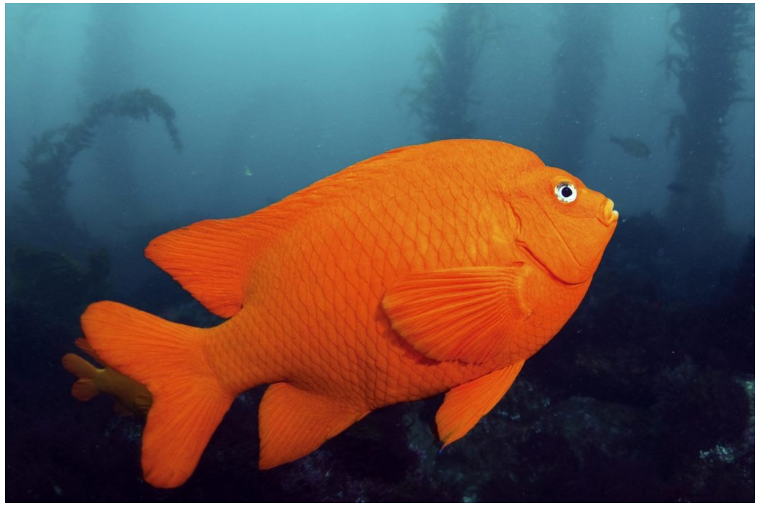

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Let $\mathbb{F}$ be a field, and let X, Y, Z be $\mathbb{F}$-vector spaces of dimension at least 2.

For $T \in X \otimes Y \otimes Z$, an expression $$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z$$

is called a decomposition of T into product tensors

$$\mathrm{rank}(T) := \min\{n : \text{there exists a decomposition of } T \text{ into } n \text{ product tensors}\}$$

Uniqueness of tensor decompositions

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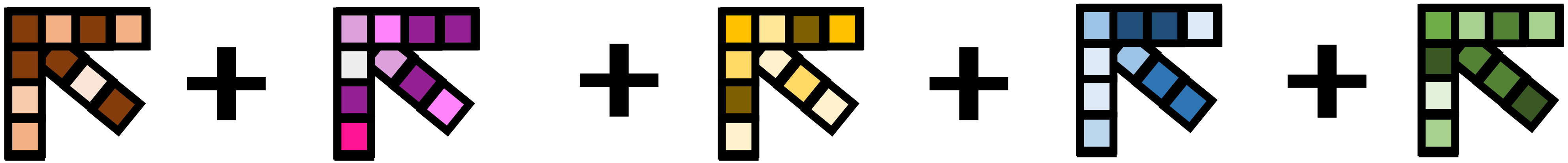

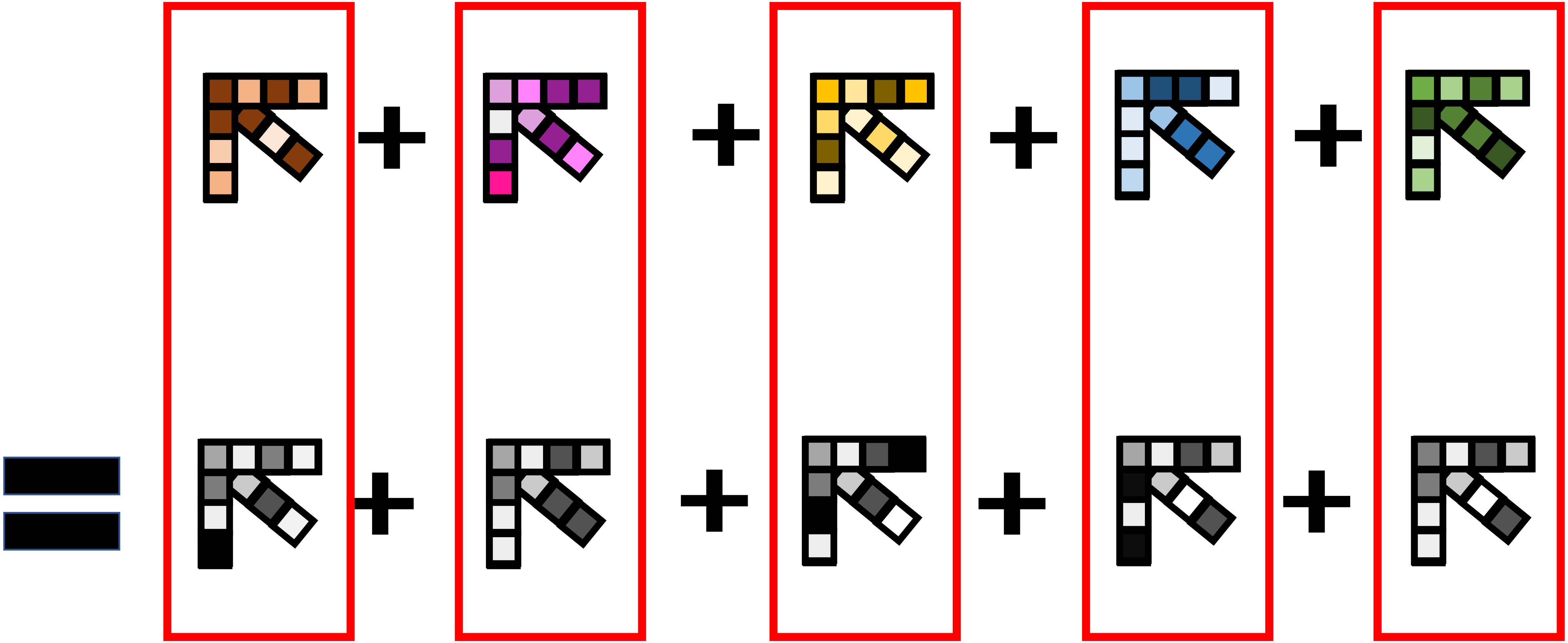

Applications

- Tensors ⟷ Physical data
- Tensor decomposition ⟷ Interpretation of data
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Blind Signal Separation

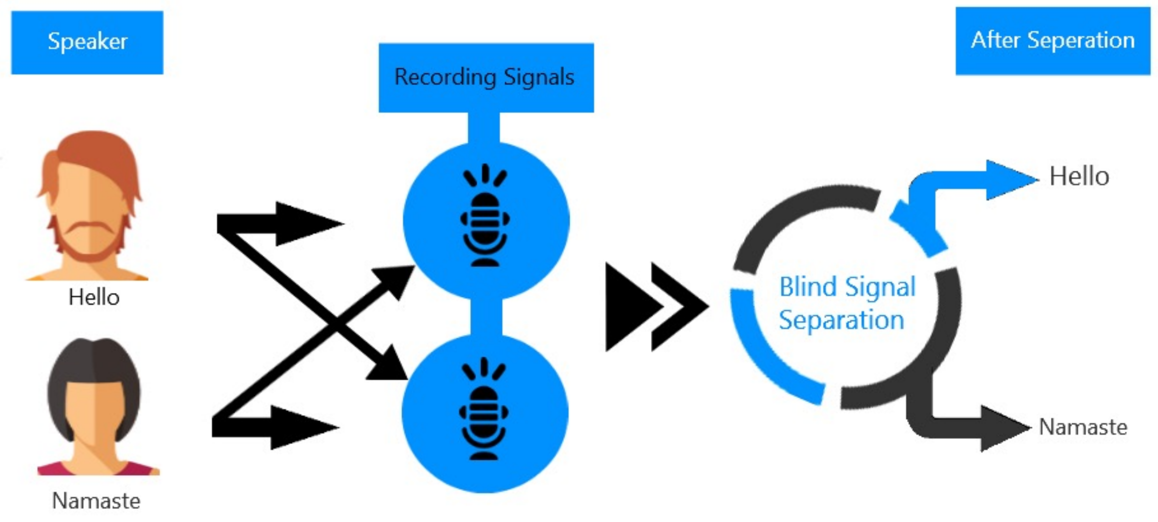

Image source: https://www.pinterest.ca/pin/66709638217102669/

Goal: Separate mixed signals + determine mixing process

Method: Decompose tensor arising from measurement data

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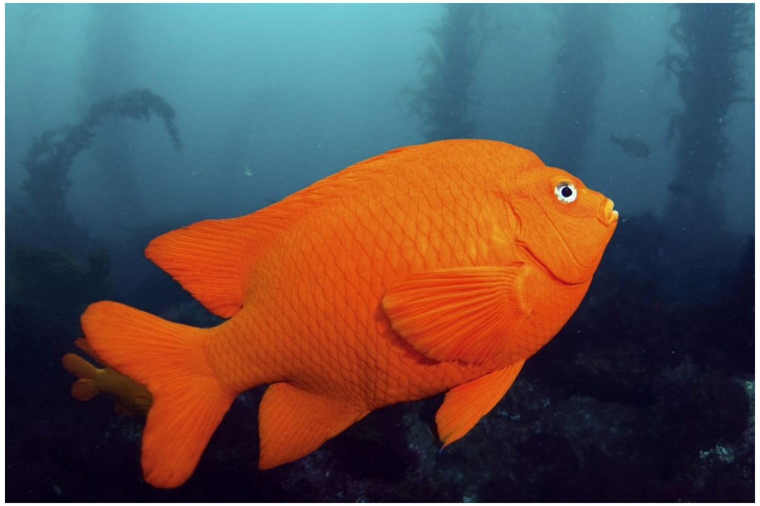

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$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

Notation: When $S \subseteq [n]$, $\quad d_x^S := \mathrm{dimspan}\{x_a : a \in S\}$

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Example:

$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} + e_4^{\otimes 3} + (e_1 + e_2) \otimes (e_2 + e_3) \otimes (e_1 + e_4)$$

$x_1 \otimes y_1 \otimes z_1$ … x_5 $\otimes$ y_5 $\otimes$ z_5

For $S = \{1,2,5\}$, $\quad d_x^S = 2,\ d_y^S = 3,\ d_z^S = 3$

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Definition: The Kruskal rank of $\{x_1, \ldots, x_n\} \in X$ is the largest integer k_x such that for every subset $S \subseteq [n]$ of size $|S| = k_x$, it holds that

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$$\{x_1, \ldots, x_5\} = \{e_1, e_2, e_3, e_4, e_1 + e_2\}, \qquad k_x = 2,\ d_x^{[5]} = 4.$$

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Example [Jennrich’s Theorem]:

$\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are linearly independent, and $k_z \geq 2$.

$k_x = k_y = n$

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Weakness: If T is unique under Kruskal’s theorem, then so is

$$T' = \sum_{a \in [n]} x_{\sigma(a)} \otimes y_a \otimes z_a \in X \otimes Y \otimes Z$$

for any $\sigma \in S_n$.

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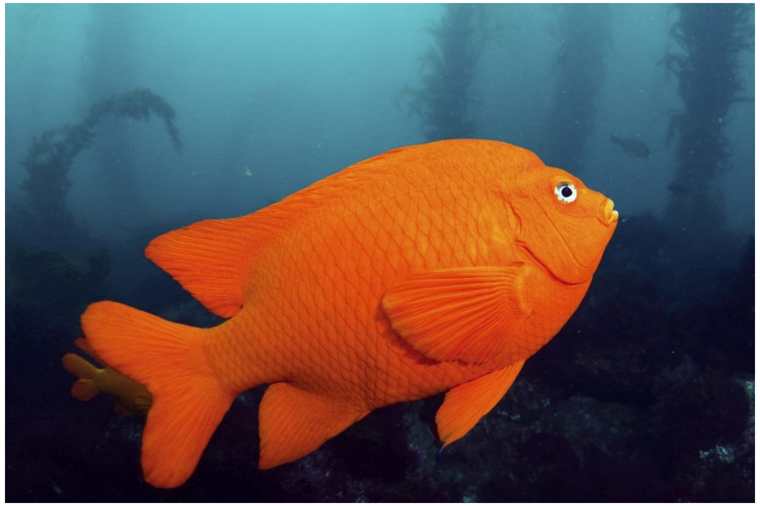

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Our generalization

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

Idea: Replace Kruskal ranks k_x with standard dimspans $d_x^{[n]}$.

<u>Theorem [Gubkin-L-Petrov]</u>: If

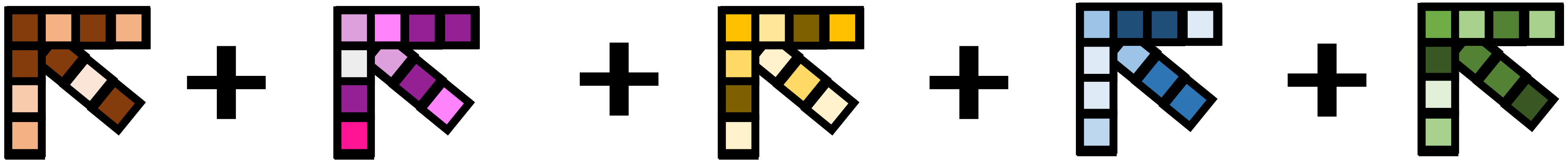
$d_x^{[n]} = \mathrm{dimspan}\{x_1, \dots, x_n\}$

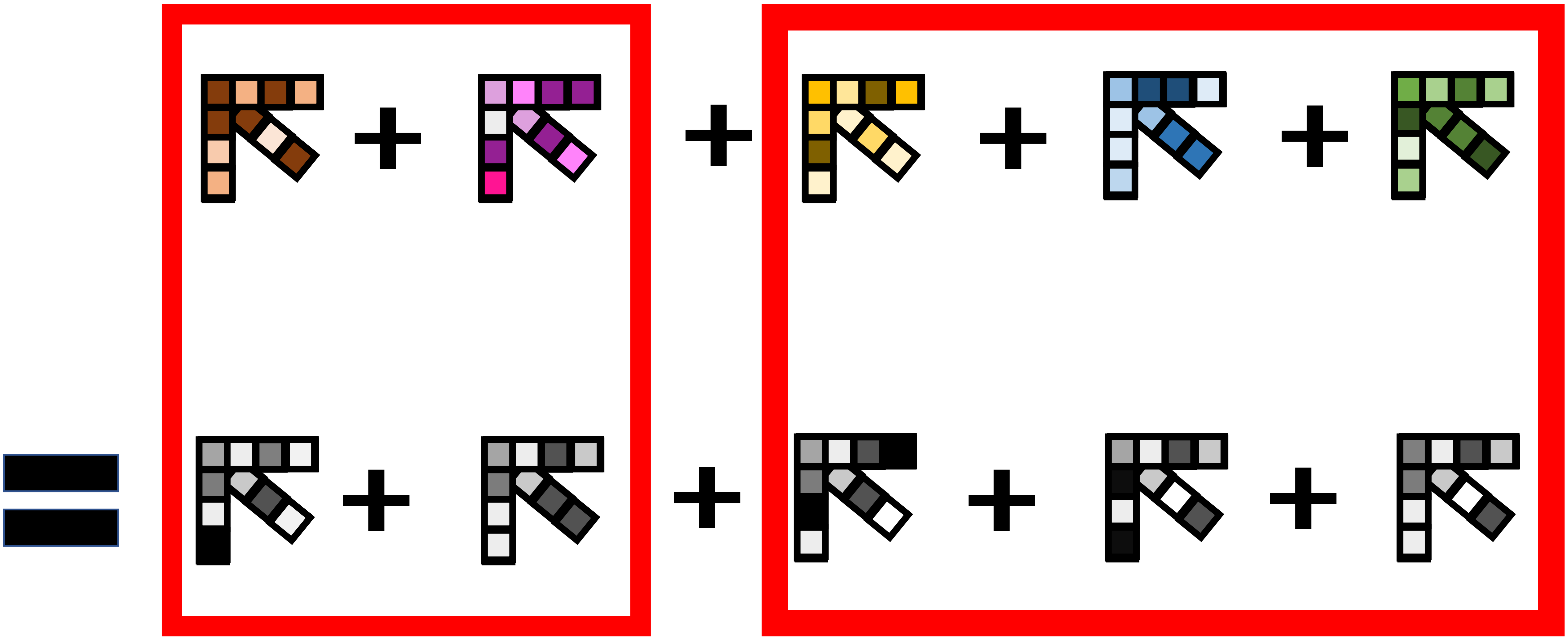
$$2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then for any other decomposition $T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a \in X \otimes Y \otimes Z$

there exist non-trivial subsets $S, R \subseteq [n]$ such that

$$\sum_{a \in S} x_a \otimes y_a \otimes z_a = \sum_{a \in R} x'_a \otimes y'_a \otimes z'_a$$





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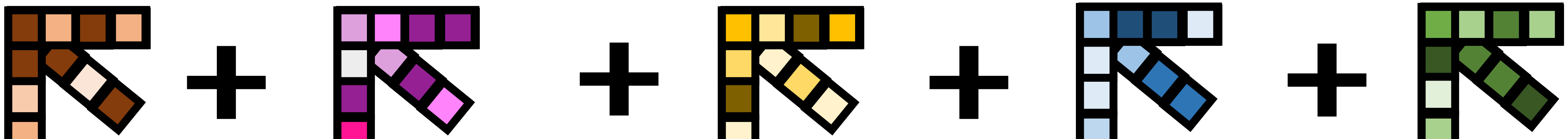

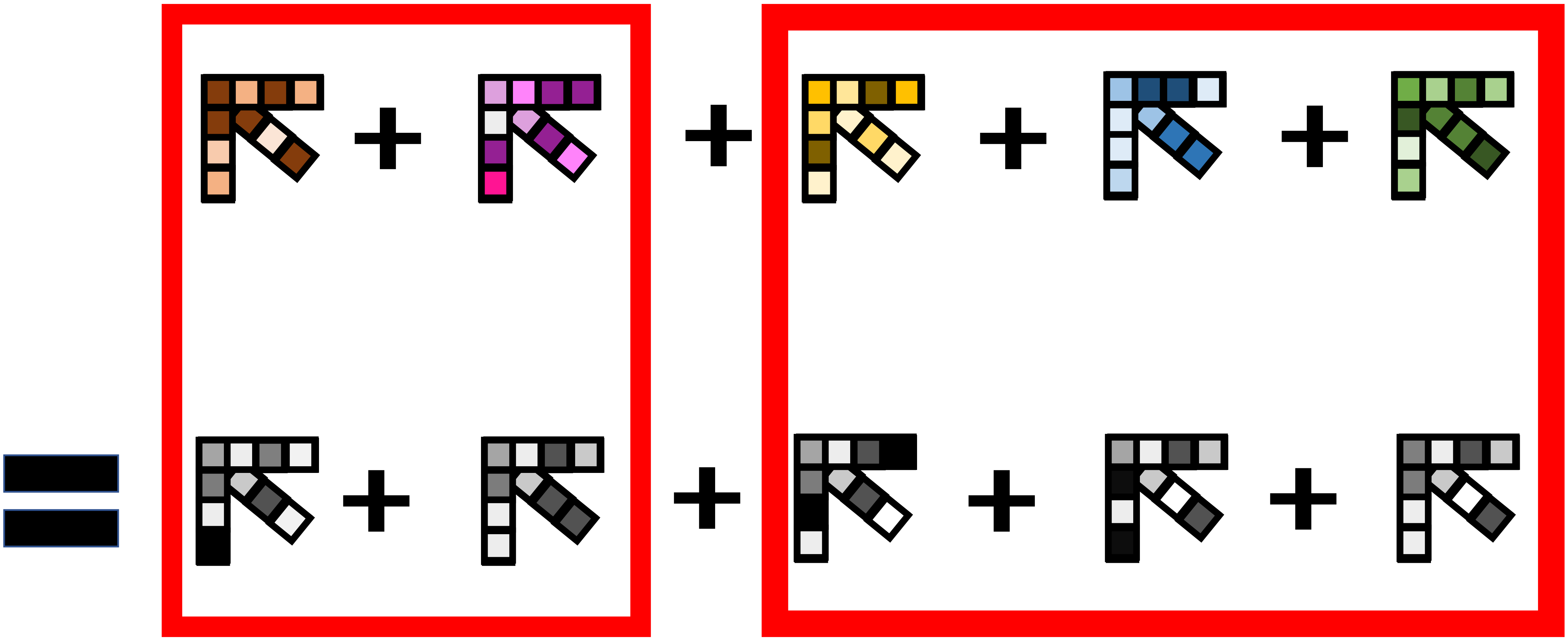

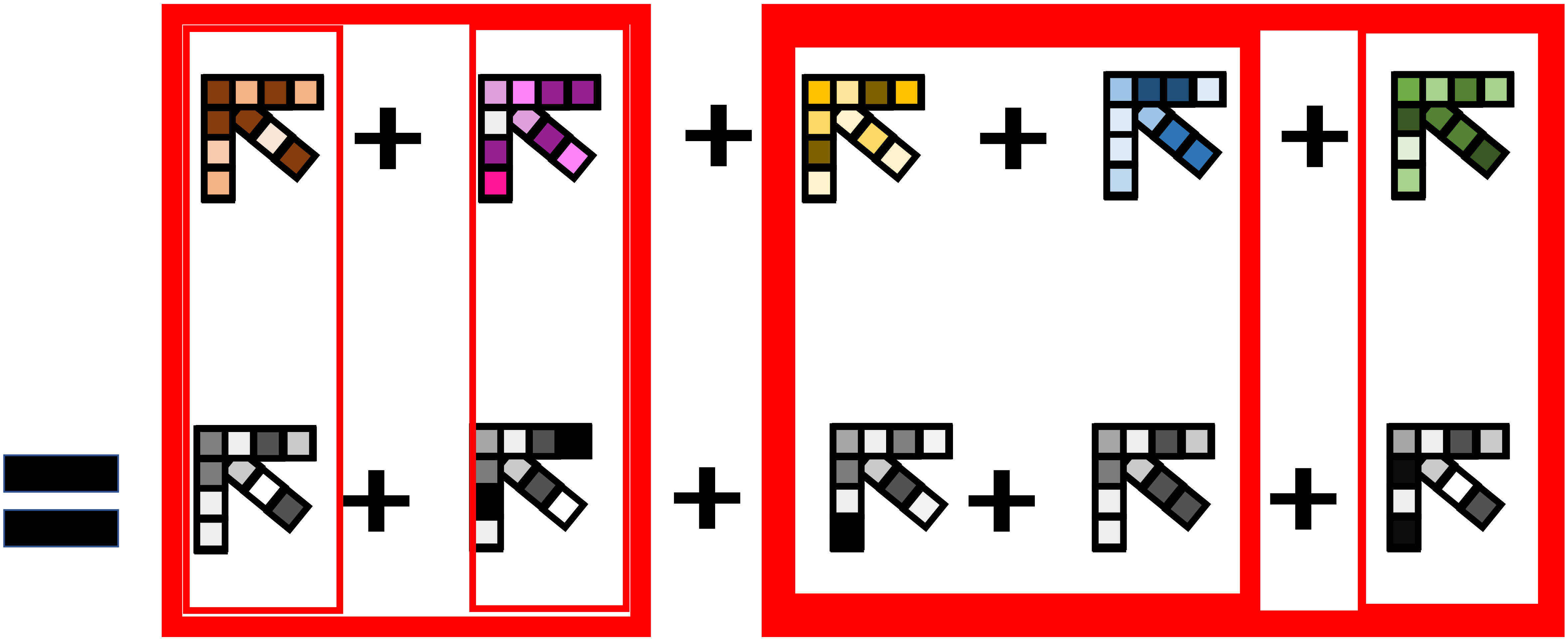

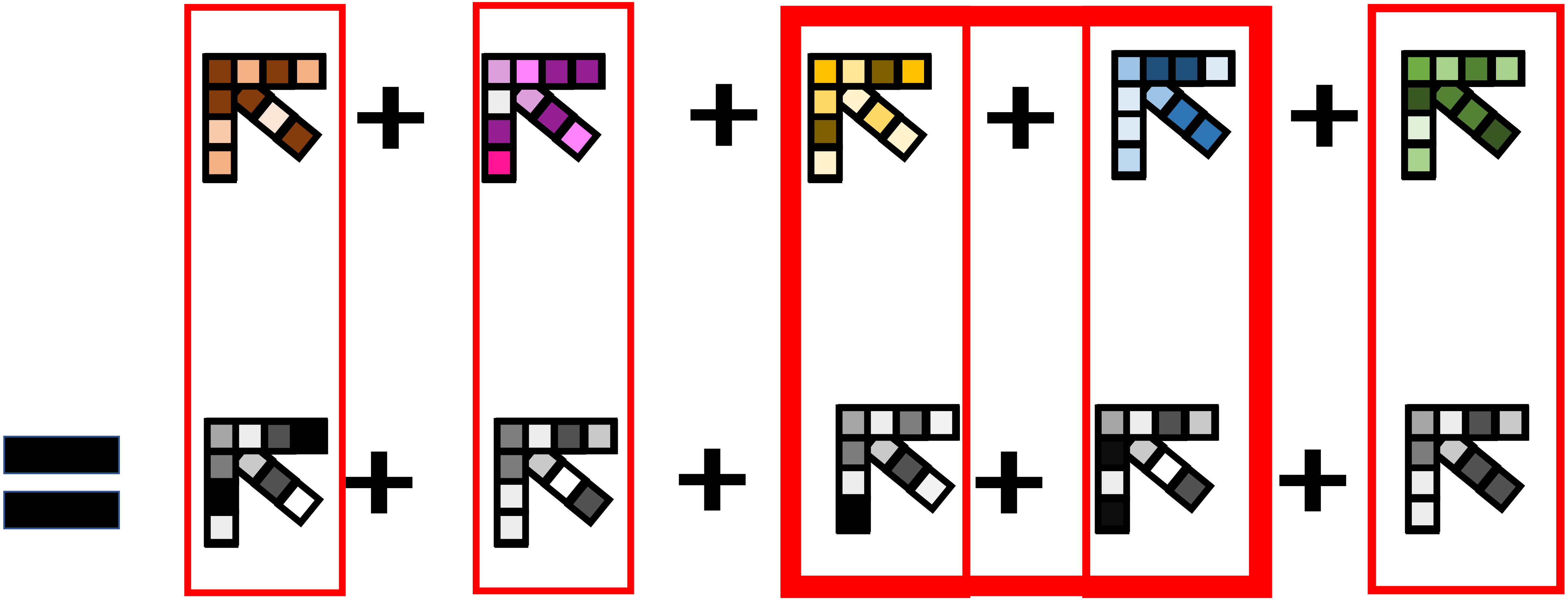

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Kruskal's theorem does not certify uniqueness

$$10 = 2n \nleq k_x + k_y + k_z - 2 = 2 + 2 + 2 - 2 = 4$$

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$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} + e_4^{\otimes 3} + (e_1 + e_2) \otimes (e_2 + e_3) \otimes (e_1 + e_4)$$

$x_1 \otimes y_1 \otimes z_1$ … x_5 $\otimes$ y_5 $\otimes$ z_5

<u>Theorem [Gubkin-L-Petrov]:</u> If for every subset $S \subseteq [n]$ of size $|S| \geq 2$, it holds that $2|S| \leq d_x^S + d_y^S + d_z^S - 2$, then (1) is the unique decomposition of T.

$10 = 2|[n]| \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2 = 4 + 4 + 4 - 2 = 10$ ✓

For $S = \{1,2\}$, $4 = 2|S| \leq d_x^S + d_y^S + d_z^S - 2 = 2 + 2 + 2 - 2 = 4$ ✓

For $S = \{1,2,5\}$, $6 = 2|S| \leq d_x^S + d_y^S + d_z^S - 2 = 2 + 3 + 3 - 2 = 6$ ✓

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By symmetric arguments for other $S \subseteq [n]$, this is the unique decomposition of T

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$x_1 \otimes y_1 \otimes z_1$ … x_5 $\otimes$ y_5 $\otimes$ z_5

Unique by [Gubkin-L-Petrov], but

$$T' = \sum_{a \in [5]} x_{\sigma(a)} \otimes y_a \otimes z_a$$

not unique for $\sigma = (13) \in S_5$

Other uniqueness results

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

1. [Domanov-Sørensen-De Lathauwer, D-De L, S-De L]
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Fact [D-De L]: If Condition U holds, then $\min\{k_y, k_z\} \geq n - d_x^{[n]} + 2$.

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Theorem: If Condition U holds, and any one of the following conditions hold then (1) is the unique decomposition of T.

1. $$k_1 + \min\{k_2, k_3 - 1\} \geq n + 1.$$
2. *It holds that $k_2 \geq 2$ and for all $\alpha \in \mathbb{F}^n$,*
   $$\operatorname{rank}\Big[\sum_{a\in[n]} \alpha_a x_{a,1} \otimes x_{a,3}\Big] \geq \min\{\omega(\alpha), n - d_2 + 2\}.$$
   (Note that this is just Condition U with the first subsystem replaced by the second).
3. *There exists a subset $S \subseteq [n]$ with $0 \leq |S| \leq d_1$ such that the following three conditions hold:*
   (a)

   40

   $$d_1^S = |S|.$$
   (b) $$d_2^{[n]\setminus S} = n - |S|.$$
   (c) *For any linear map $\Pi \in \mathrm{L}(\mathcal{V}_1)$ with $\ker(\Pi) = \operatorname{span}\{x_{a,1} : a \in S\}$, scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, and index $b \in [n] \setminus S$ such that*
   $$\sum_{a\in[n]\setminus S} \alpha_a \Pi x_{a,1} \otimes x_{a,3} = \Pi x_{b,1} \otimes z$$
   for some $z \in \mathcal{V}_{\sigma(3)}$, it holds that $\omega(\alpha) \leq 1$.
4. *There exists a permutation $\tau \in S_n$ for which the matrix*
   $$X_1^\tau = (x_{\tau(1),1}, \ldots, x_{\tau(n),n})$$
   has reduced row echelon form
   $$Y = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & Z \\ & & 1 & \end{array}\right],$$
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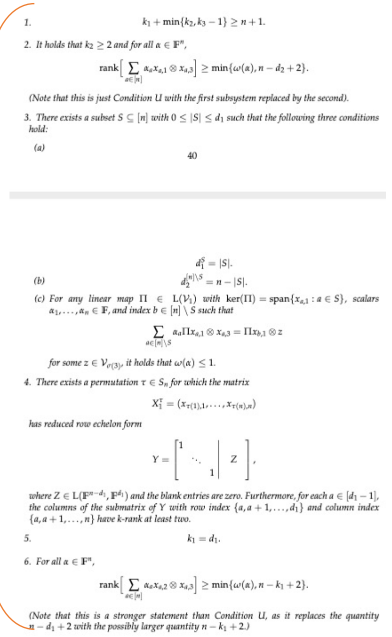

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Proof sketch: By Kruskal's permutation lemma, Condition U implies that the x_a's are unique. Use extra assumptions to prove full uniqueness.

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Summary [Gubkin-L-Petrov]:

Condition U $\Longrightarrow$ Unique

$\not\Uparrow \; \not\Downarrow$

$2|S| \leq d_x^S + d_y^S + d_z^S - 2$ for all $S \subseteq [n]$ s.t $|S| \geq 2$ $\Longrightarrow$ Unique

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Related: Uniqueness of symmetric decompositions, generic uniqueness, uniqueness for low rank tensors, finite decompositions,…

Outline

- Uniqueness of tensor decompositions
- Kruskal's theorem
- A generalization of Kruskal's theorem
- Splitting theorem

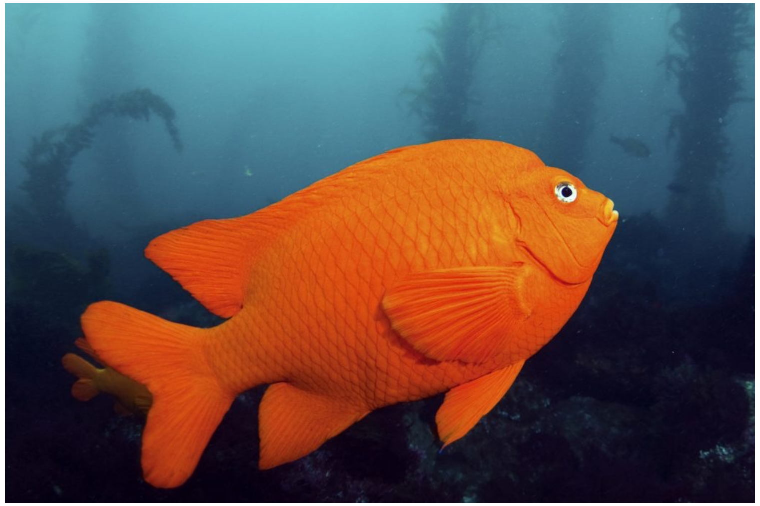

Image: https://fishingbooker.com/blog/california-state-fish-golden-trout-garibaldi/

Splitting

Definition: A set of vectors $E = \{v_1, \dots, v_n\}$ **splits** if there exists a non-trivial subset $S \subseteq E$ such that

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$$\Leftrightarrow \mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$$

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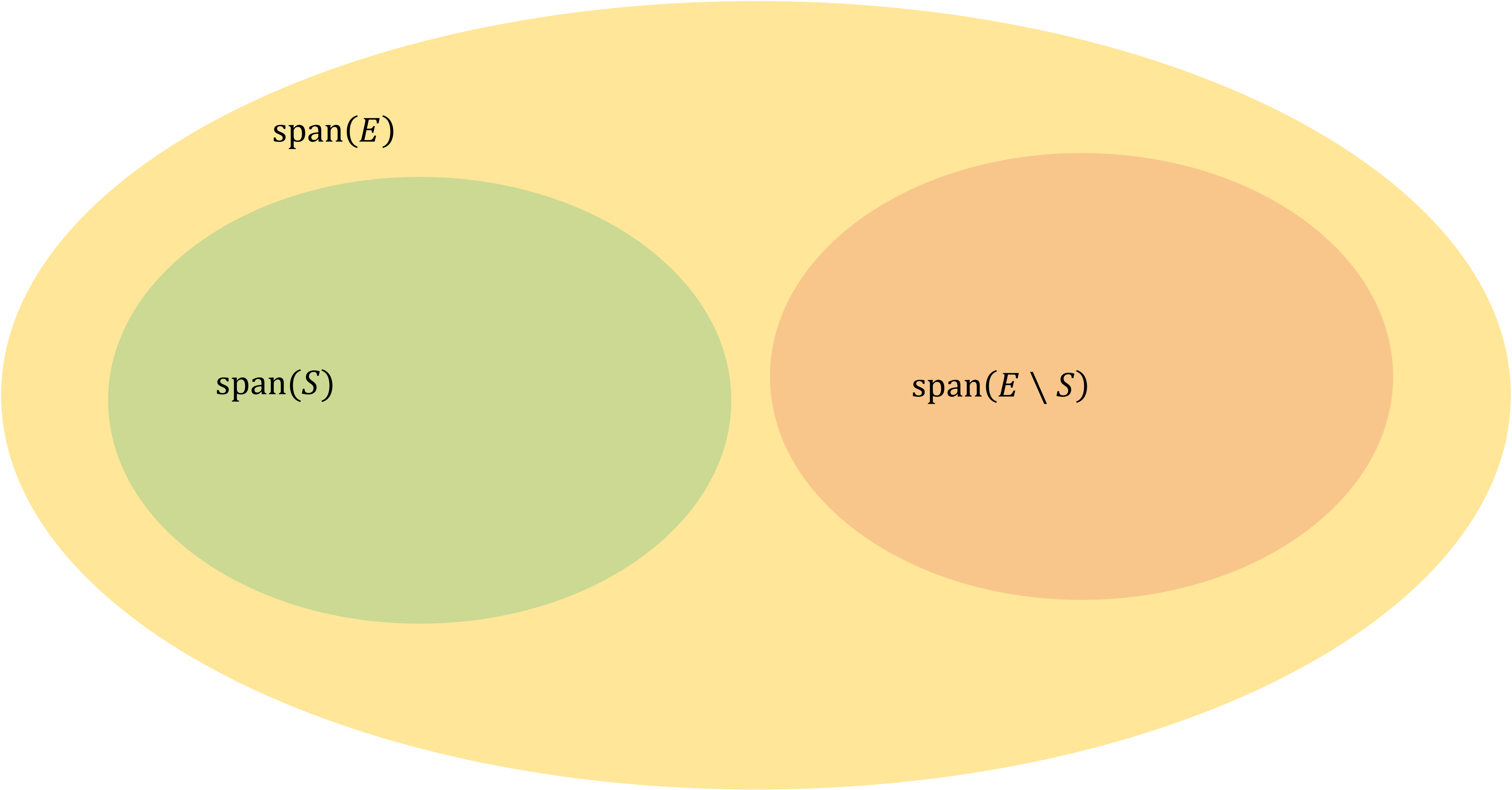

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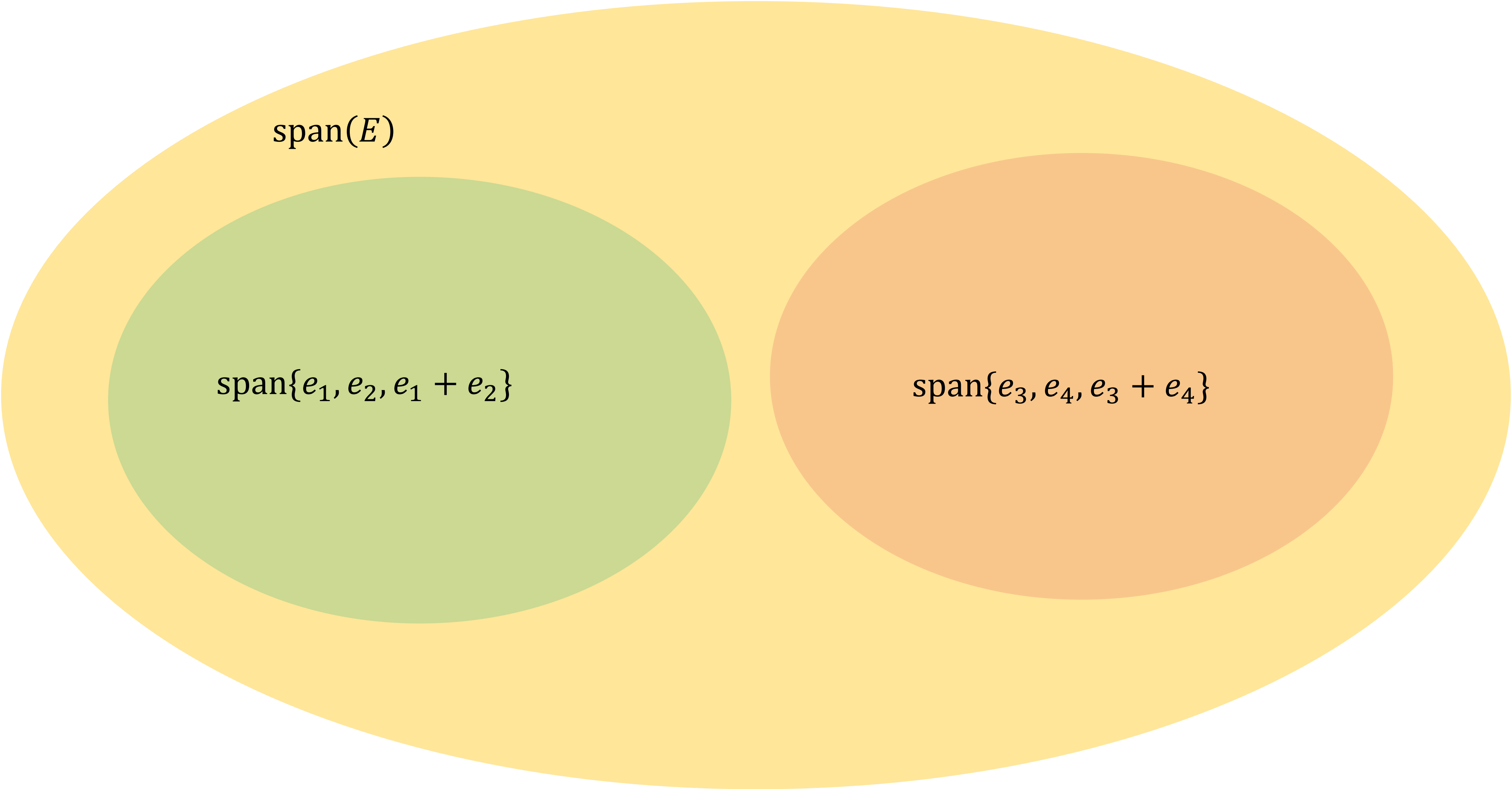

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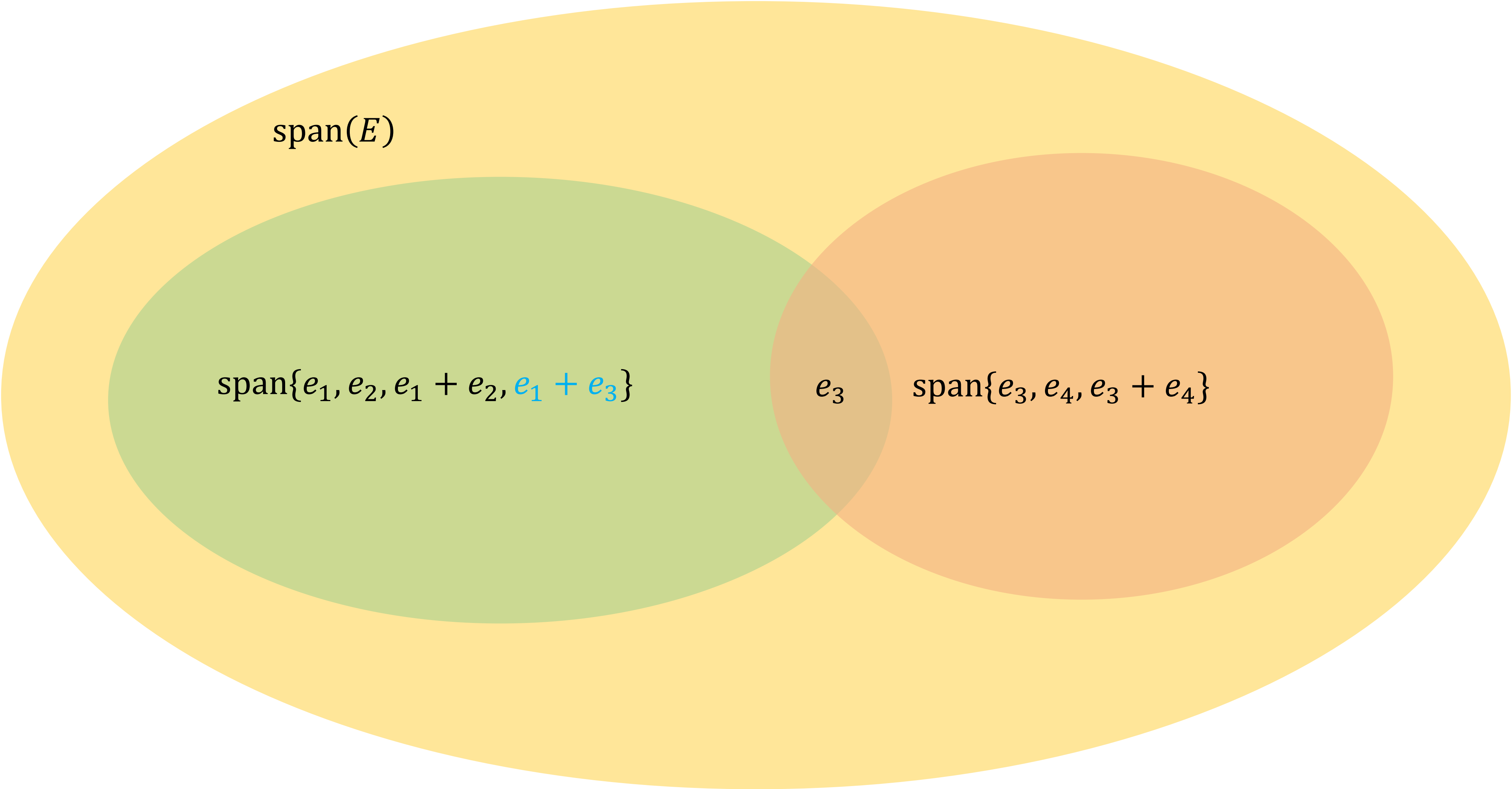

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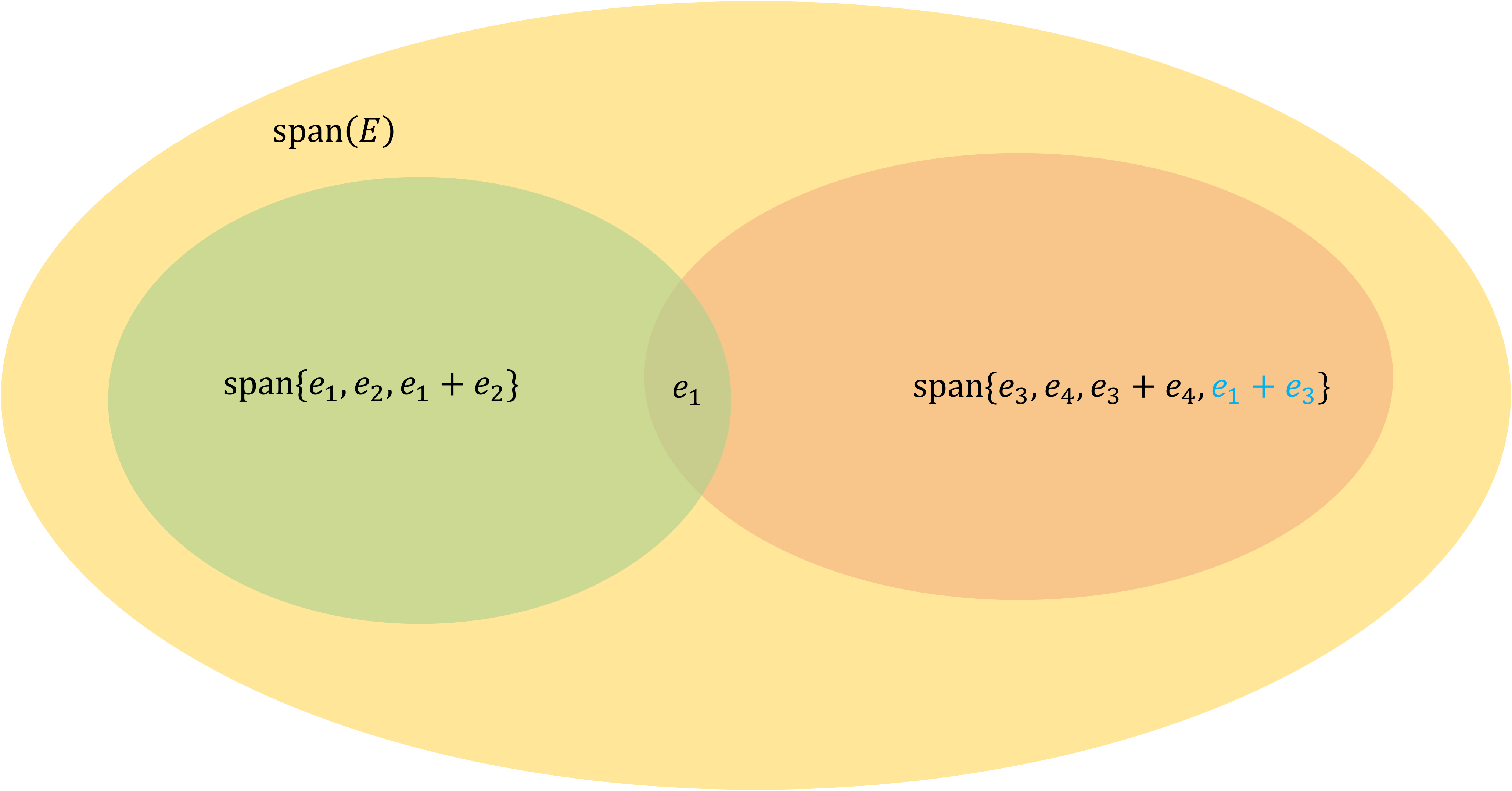

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Proof: $\sum(E) = 0 \Rightarrow \sum(S) = -\sum(E \setminus S) \in \mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$ ∎

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(Our generalization of Kruskal's theorem is a corollary to this)

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Sylvester's rank inequality: $n \leq d_x^{[n]} + d_y^{[n]} - \mathrm{rank}(XY^T)$

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Example: $E = \{e_1 \otimes e_1, \quad e_2 \otimes e_2, \quad e_3 \otimes e_2, \quad (e_2+e_3) \otimes e_2\}$ splits.

$$4 = n \leq d_x^{[4]} + d_y^{[4]} - 1 = 3 + 2 - 1 = 4$$

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Splitting theorem => Corollary: If E is linearly independent, then it splits.

Otherwise, $\mathrm{dimspan}(E) \leq n - 1 \leq d_x^{[n]} + d_y^{[n]} - 2,$ so E splits by splitting theorem.

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$$\mathrm{dimspan}(E) \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 3$$

then E splits.

$d_x^{[n]} = \mathrm{dimspan}\{x_1, \dots, x_n\}$

Corollary: If

$$n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then E splits.

Splitting theorem

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(E) = \mathrm{span}(S) \oplus \mathrm{span}(E \setminus S)$

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Corollary: If

$$n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then E splits.

Corollary: If

$$2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then for any other set of product tensors $E' = \{x_a' \otimes y_a' \otimes z_a' : a \in [n]\}$, $E \cup E'$ splits.

Corollary => Kruskal generalization

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

Suffices to prove:

Theorem [Gubkin-L-Petrov]: If

$d_x^{[n]} = \text{dimspan}\{x_1, \ldots, x_n\}$

$$2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then for any other decomposition $T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a \in X \otimes Y \otimes Z$

there exist non-trivial subsets $S, R \subseteq [n]$ such that $\sum_{a \in S} x_a \otimes y_a \otimes z_a = \sum_{a \in R} x'_a \otimes y'_a \otimes z'_a$

Corollary => Kruskal generalization

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

Suffices to prove:

<u>Theorem [Gubkin-L-Petrov]:</u> If

$d_x^{[n]} = \mathrm{dimspan}\{x_1, \dots, x_n\}$

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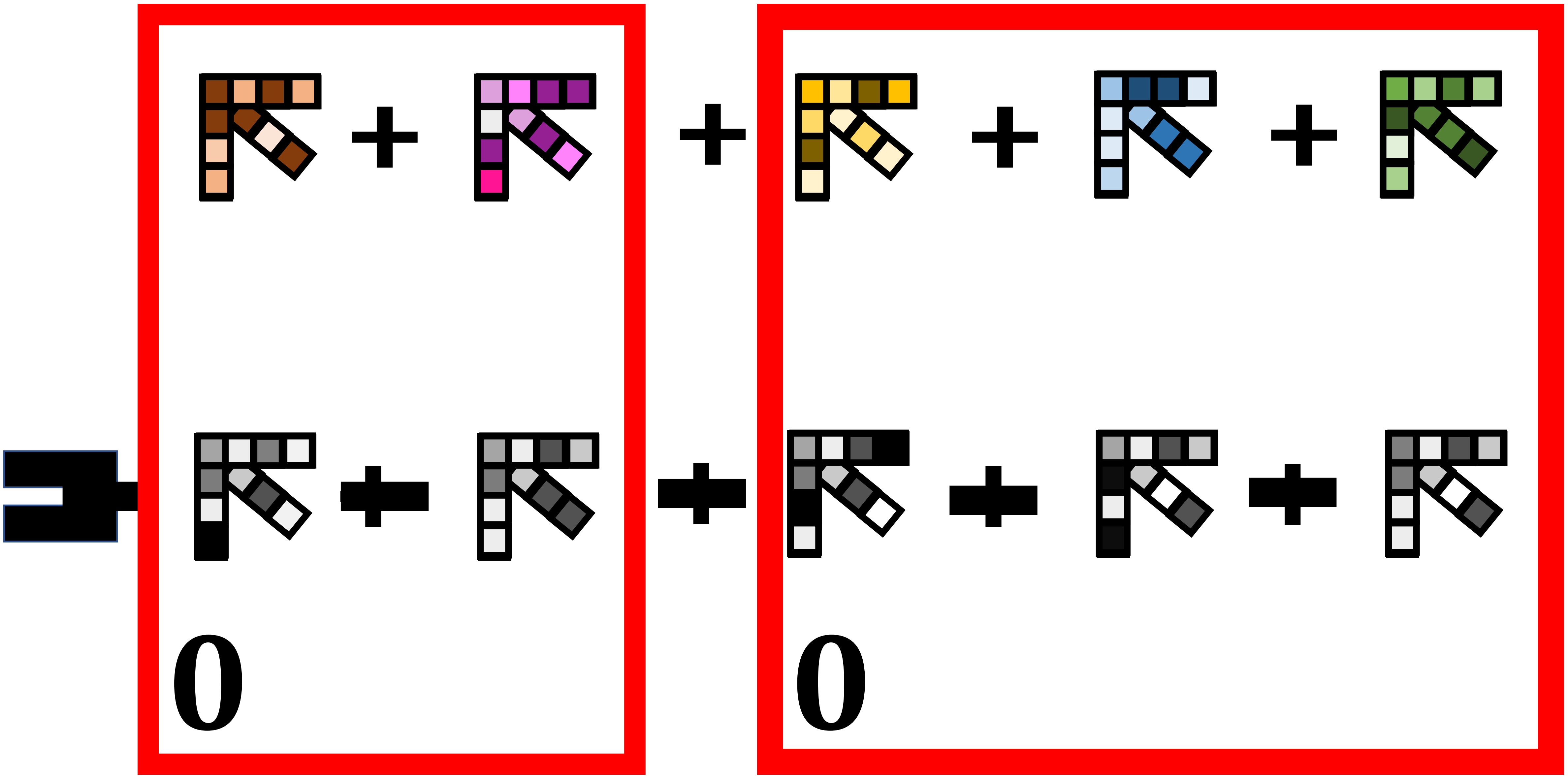
Proof:

By previous corollary, $\{x_a \otimes y_a \otimes z_a, x'_a \otimes y'_a \otimes z'_a \ : a \in [n]\}$ splits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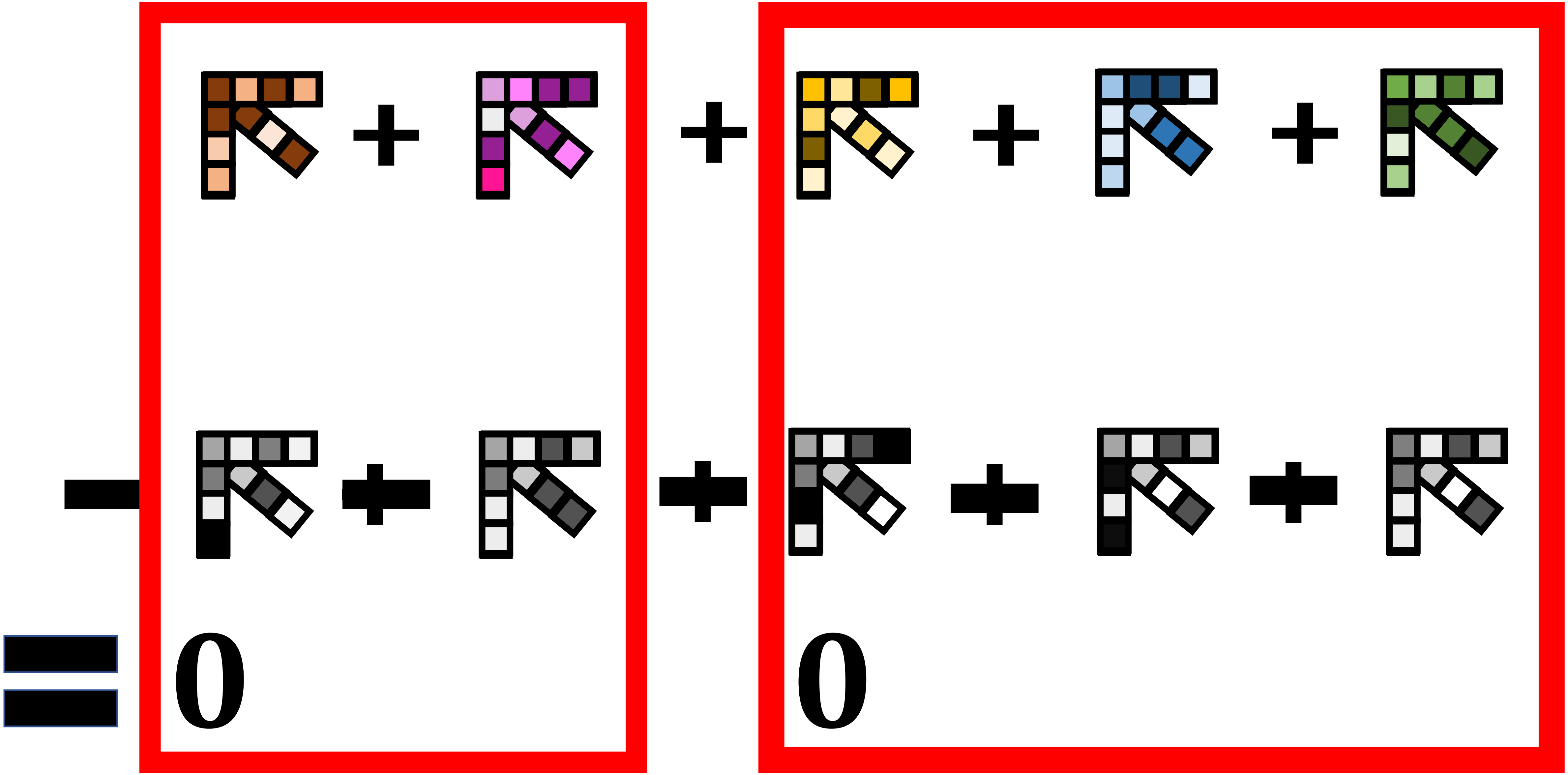
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Conclusion

- Uniqueness of tensor decompositions
- Kruskal's theorem
- A generalization of Kruskal's theorem
- Splitting theorem
- More matroid theory for product tensors?

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