The convex set of k-incoherent gram matrices

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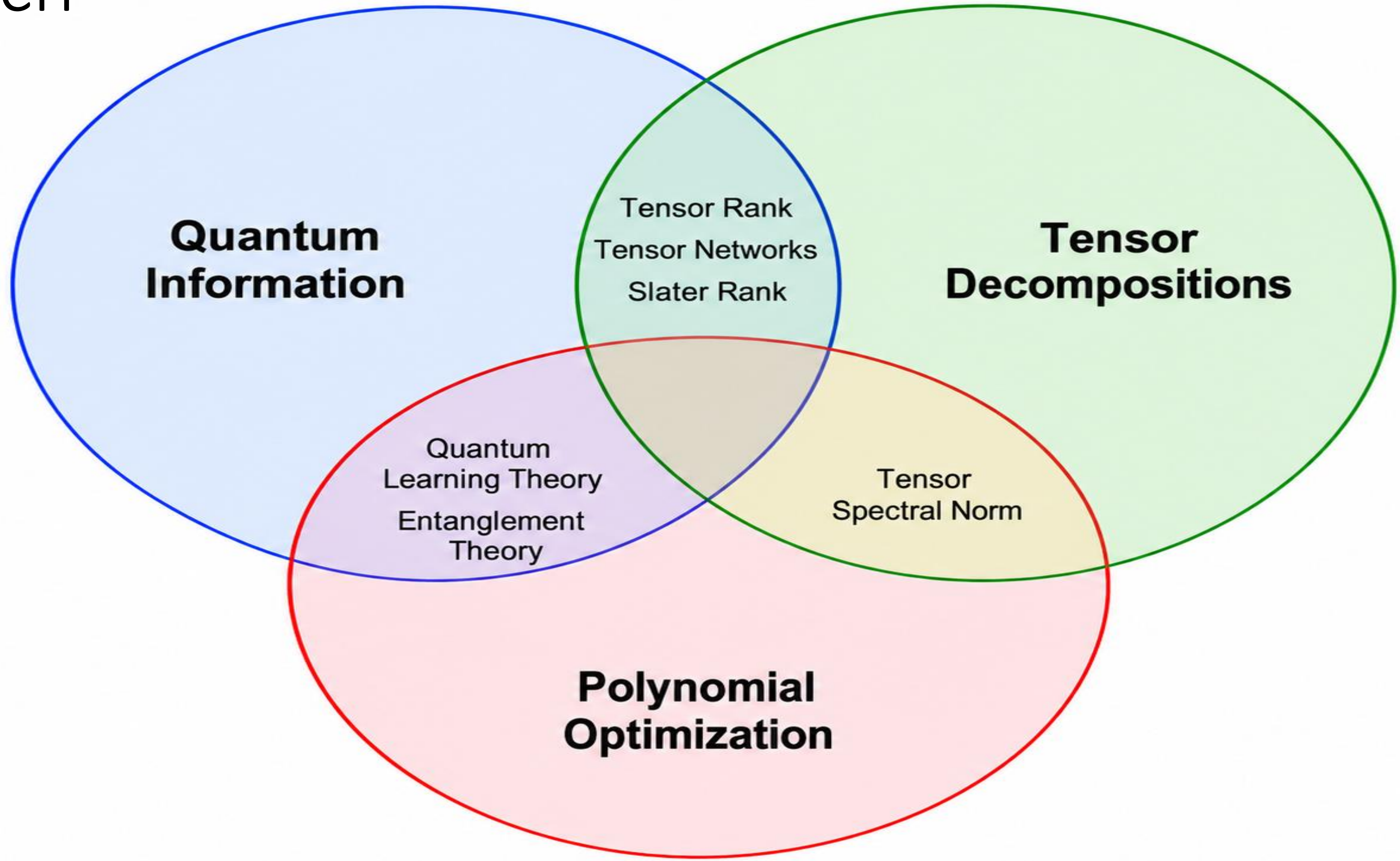
My research


Quantum
Information
Tensor Rank
Tensor Networks
Slater Rank
Tensor
Decompositions
Quantum
Learning Theory
Entanglement
Theory
Tensor
Spectral Norm
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Optimization

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Mathematical Programming
https://doi.org/10.1007/s10107-025-02251-y

FULL LENGTH PAPER

Series A

A hierarchy of eigencomputations for polynomial optimization on the sphere

Benjamin Lovitz[1] · Nathaniel Johnston[2]

Received: 3 December 2024 / Accepted: 21 June 2025

Goal:

Given $p \in \mathbb{R}[x_1, \ldots, x_n]_{2d}$

compute $p_* := \min_{\|x\|=1} p(x)$

RSOS hierarchy [Reznick, Lasserre]:

$p_* > 0 \;\; \Rightarrow \;\; p_k(x) := p(x)\|x\|^{2(k-d)} \in SOS$ for some k

Convergence [Fang-Fawzi 19]:

$$p_k \notin SOS \Rightarrow p_* \lesssim 1/k^2$$

Problem with RSOS: Checking containment in *SOS* requires semidefinite programming (slow).

- More details: Requires searching for PSD Gram matrix in affine subspace of matrix rep's of $p_k(x)$.
- [Lovitz-Johnston]: Suffices to check PSD-ness of a **single canonical** Gram matrix (eigencomputation) (HSOS)
- Downside: Convergence is now $\approx 1/k$

Trade-off: RSOS hierarchy has fast convergence $1/k^2$, but each level is slow to compute
Lovitz-Johnston hierarchy has slow convergence $1/k$, but each level is fast to compute

Ad for a different paper

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A hierarchy of eigencomputations for polynomial optimization on the sphere

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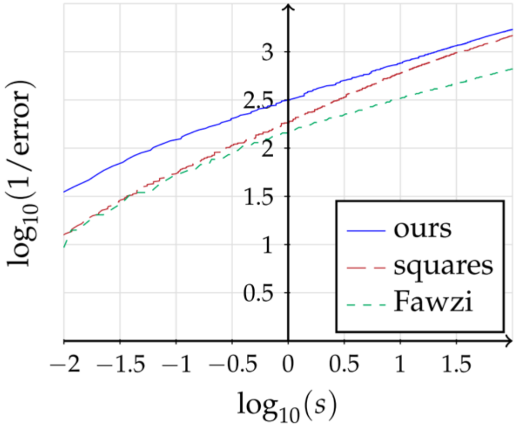
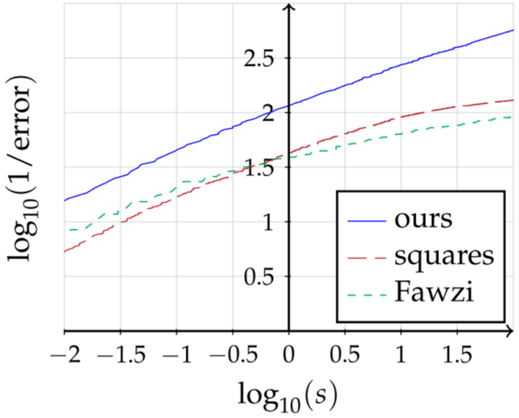
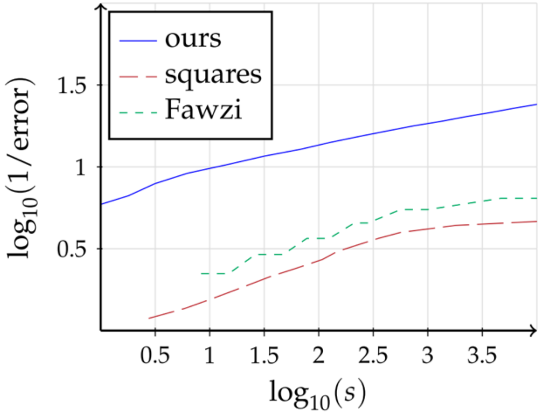
So… What does the trade-off look like in practice?



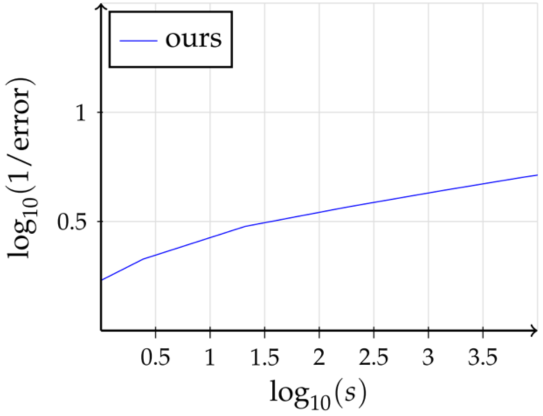
(a) The Motzkin polynomial.



(b) A random dense 3-variable quartic.



(c) A random dense 6-variable quartic.



(d) A random dense 10-variable quartic.

The actual talk (please silence your cellphones now)

Rest of talk based on:

arXiv > quant-ph > arXiv:2510.20789

Quantum Physics

[Submitted on 23 Oct 2025]

The complexity of perfect quantum state classification

Nathaniel Johnston, Benjamin Lovitz, Vincent Russo, Jamie Sikora

+ Forthcoming work by same authors

Quantum computing 101

Unknown quantum state (unit vector) $|\psi\rangle \in \mathbb{C}^d$ in q.c.

Obtain info about $|\psi\rangle$ by applying a *measurement*.

Measurement: A set of positive semidefinite $d \times d$ matrices $M_1, \ldots, M_\ell$ such that $\sum_i M_i = I_d$

Result: Outcome i with probability $p_i = \langle\psi|M_i|\psi\rangle$

… and $|\psi\rangle$ is destroyed.

Problem: Cannot learn much about $|\psi\rangle$ from a single measurement (uncertainty principle).

Problem: We want to obtain information about $|\psi\rangle \in \mathbb{C}^d$, but a single measurement gives little information about $|\psi\rangle$.

Solution: Make prior assumptions about $|\psi\rangle$

and seek only limited information.

(Quantum learning theory)

The k-learnability problem

Let $|\psi_1\rangle, \dots, |\psi_n\rangle \in \mathbb{C}^d$ be fixed, known unit vectors.

We are handed an unknown quantum state $|\psi\rangle$ and promised that $|\psi\rangle = |\psi_j\rangle$ for some j.

Goal (1-learning): Determine j (no errors allowed)

Example: Suppose $n = d$ and $|\psi_i\rangle = |i\rangle$ is the i-th standard basis vector.

Let $M_i = |i\rangle\langle i|$. This is a measurement because $M_i \geq 0$ and $\sum_i M_i = I_d$.

Furthermore, $p_i = \langle j|M_i|j\rangle = \delta_{i,j}$. So the measurement output is j !

In general, a collection $|\psi_1\rangle, \dots, |\psi_n\rangle \in \mathbb{C}^d$ is 1-learnable $\Longleftrightarrow$ they are orthogonal

K-learning

The k-learnability problem

Let $|\psi_1\rangle, \ldots, |\psi_n\rangle \in \mathbb{C}^d$ be fixed, known unit vectors.

We are handed an unknown quantum state $|\psi\rangle$ and promised that $|\psi\rangle = |\psi_j\rangle$ for some j.

Goal (k-learning): Determine a k-element subset $S \subseteq \{1, \ldots, n\}$ for which $j \in S$.

In other words, we want to find positive operators $\{M_S \ : S \subseteq [n], \ \ |S| = k\}$ s.t. $\sum_S M_S = I_d$ and $\langle\psi_i|M_S|\psi_i\rangle = 0$ whenever $i \notin S$.

But there is a more convenient formulation…

The k-learnability problem

Let $|\psi_1\rangle, \dots, |\psi_n\rangle \in \mathbb{C}^d$ be fixed, known unit vectors.

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Goal (k-learning): Determine a k-element subset $S \subseteq \{1, \dots, n\}$ for which $j \in S$.

Def: Gram matrix of $|\psi_1\rangle, \dots, |\psi_n\rangle$ is defined as $G := (\langle\psi_i|\psi_j\rangle)_{i,j} \in \mathrm{Mat}_{n\times n}$

$$G := \begin{pmatrix} 1 & \langle\psi_1|\psi_2\rangle & \langle\psi_1|\psi_3\rangle & \cdots & \langle\psi_1|\psi_n\rangle \\ \langle\psi_2|\psi_1\rangle & 1 & \langle\psi_2|\psi_3\rangle & \cdots & \langle\psi_2|\psi_n\rangle \\ \langle\psi_3|\psi_1\rangle & \langle\psi_3|\psi_2\rangle & 1 & \cdots & \langle\psi_3|\psi_n\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle\psi_n|\psi_1\rangle & \langle\psi_n|\psi_2\rangle & \langle\psi_n|\psi_3\rangle & \cdots & 1 \end{pmatrix}$$

The k-learnability problem

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Def: G is k-incoherent if we can write $G = \sum_i P_i$, where each P_i is a PSD matrix supported on a $k \times k$ principal submatrix.

Example: $\begin{bmatrix} 1 & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a PSD matrix supported on a 2x2 principal submatrix

The k-learnability problem

Let $|\psi_1\rangle, \dots, |\psi_n\rangle \in \mathbb{C}^d$ be fixed, known unit vectors.

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Example: $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a PSD matrix supported on a 2x2 principal submatrix.

Fact [Johnston-L-Russo-Sikora 25]: $|\psi_1\rangle, \dots, |\psi_n\rangle$ is k-learnable $\iff$ G is k-incoherent

Deciding k-incoherence

PSD matrix with 1's on diagonal

Problem: Given $n \times n$ Gram matrix $G \geq 0$, decide if G is k-incoherent.

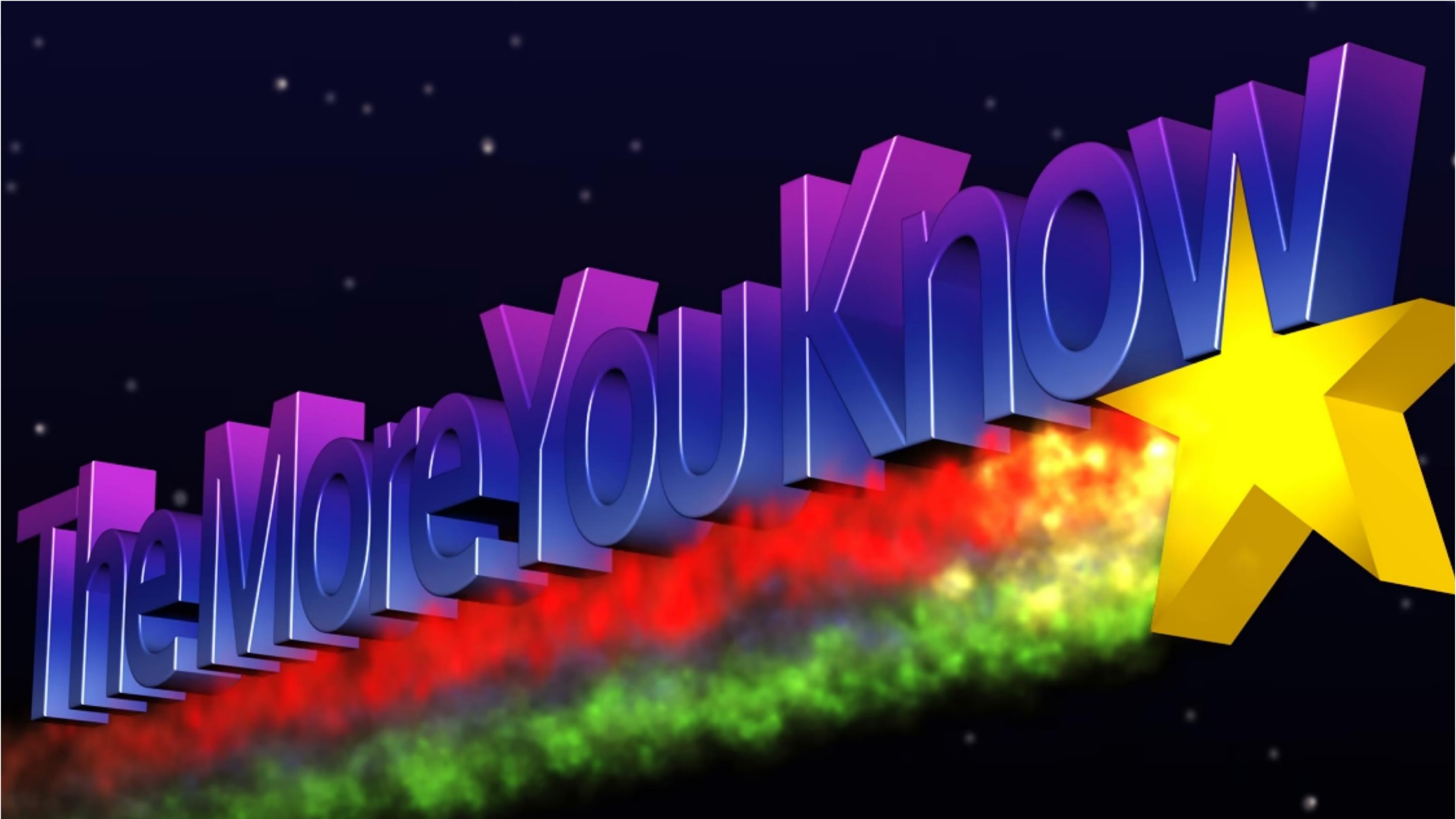
Algorithm (SDP): Determine if we can write $G = \sum_S P_S$

…where S ranges over all k-element subsets of $\{1, \dots, n\}$
and $P_S \geq 0$ is a PSD matrix supported on S.

Complexity: $\binom{n}{k}$ matrix variables $\longrightarrow$ $n^{O(k)}$-time algorithm

Theorem 1 [J-L-R-S 25]:

- There is a poly(n)-time algorithm to decide k-incoherence when k is fixed or $\mathrm{rank}(G)$ is fixed.
- If k is part of the input, deciding k-incoh is NP Hard (reduction to k-clique)


The More You Know

A sufficient condition for k-incoherence

Let G be a Gram matrix (PSD with 1's on diagonal)

Def: G is k-incoherent if we can write $G = \sum_i P_i$, where each P_i is a PSD matrix supported on a $k \times k$ principal submatrix.

Easy: G is 1-incoherent $\Leftrightarrow$ $G = I_n$ $\Leftrightarrow$ $\|G\|_F = n$

Theorem 2 [J-L-R-S 26+]: If $\|G\|_F \leq \frac{n}{\sqrt{n-k+1}}$ then G is k-incoherent.

Theorem 3 [J-L-R-S 26+]: If $\sum_{i,j} |G_{i,j}| > nk$ then G is NOT k-incoherent.

Note: Both inequalities are sharp

Packing bounds: How large can a k-learnable set in $\mathbb{C}^d$ be?

$\Leftrightarrow$ How large can a k-incoherent, rank-d Gram matrix be?

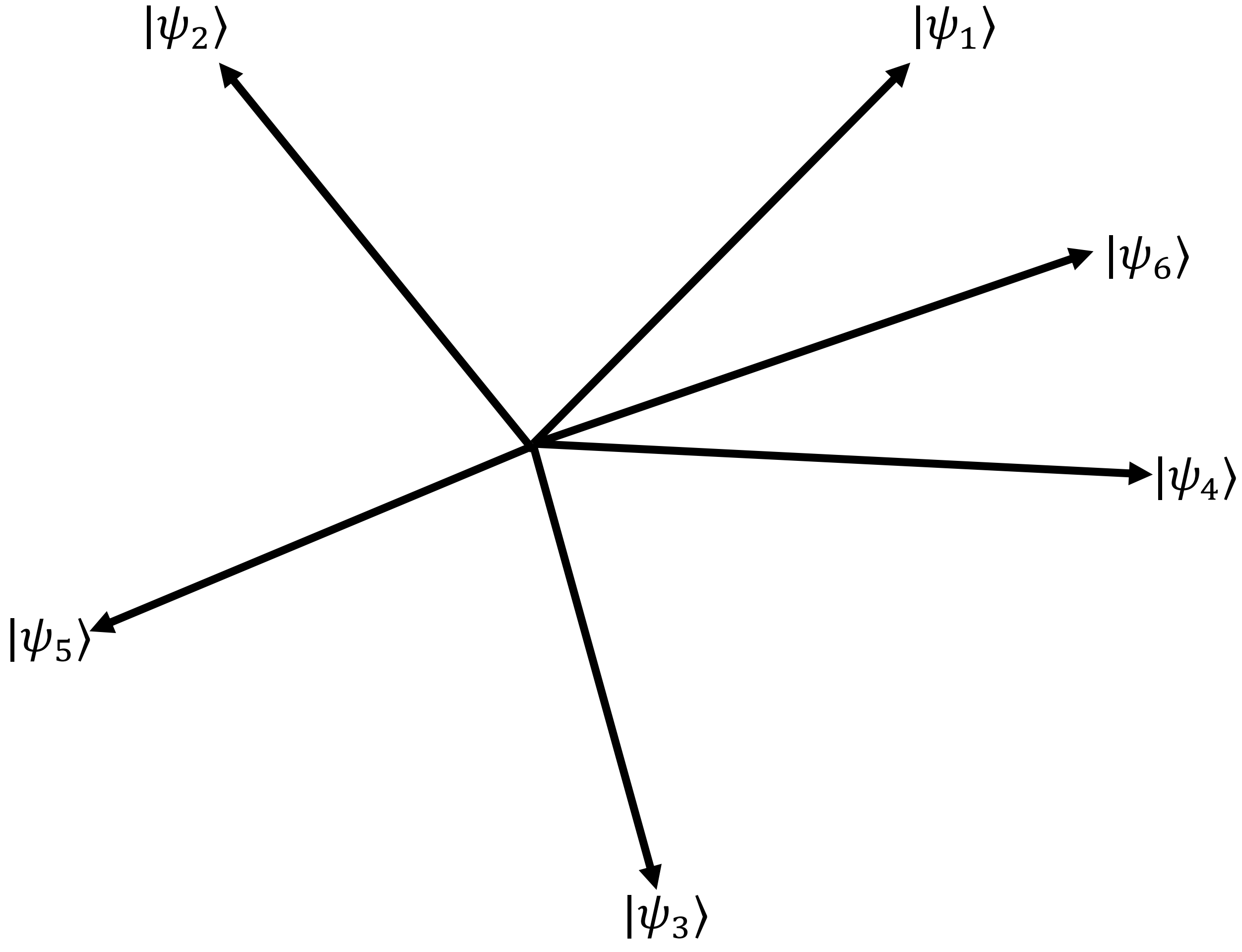
Thm 4 [JLRS 26+]: $n \leq dk$

Inequality is sharp. Attained by

$$\underbrace{|1\rangle, \ldots, |1\rangle}_{k \text{ times}}, \underbrace{|2\rangle, \ldots, |2\rangle}_{k \text{ times}}, \cdots, \underbrace{|d\rangle, \ldots, |d\rangle}_{k \text{ times}}$$

Thm 5 [JLRS 26+]:
For pairwise non-parallel vectors,
$n \leq \frac{1}{2} d(k+1)$.

Inequality is sharp.



Open questions

1. What if we are given $|\psi\rangle^{\otimes 2}$?

   - The new Gram matrix is $G \circ G = \left(\langle\psi_i|\psi_j\rangle^2\right)_{i,j}$.

2. SDP hierarchies? Convergence?

3. What about approximate k-learnability/k-incoherence?

4. What about k-learnability for mixed states?

5. Determine k-learnability of some famous sets of states (Stabilizer states, SIC-POVMs, …).

A hierarchy (HSOS) for k-incoherence

Let $S_k \subseteq \mathbb{C}^d$ be the set of k-sparse vectors, and let

$$S_k^{(n)} := \mathrm{span}\{v^{\otimes n} : v \in S_k\}$$

<u>Fact</u>: G is k-incoherent $\iff$ For all n there exists $P_n \in \mathrm{Pos}\left(\left(\mathbb{C}^d\right)^{\otimes n}\right)$

s.t. $\mathrm{im}(P_n) \subseteq S_k^{(n)}$ and $\mathrm{Tr}_{n-1}(P_n) = G$

Follows e.g. from [Derksen-Johnston-Lovitz 24].

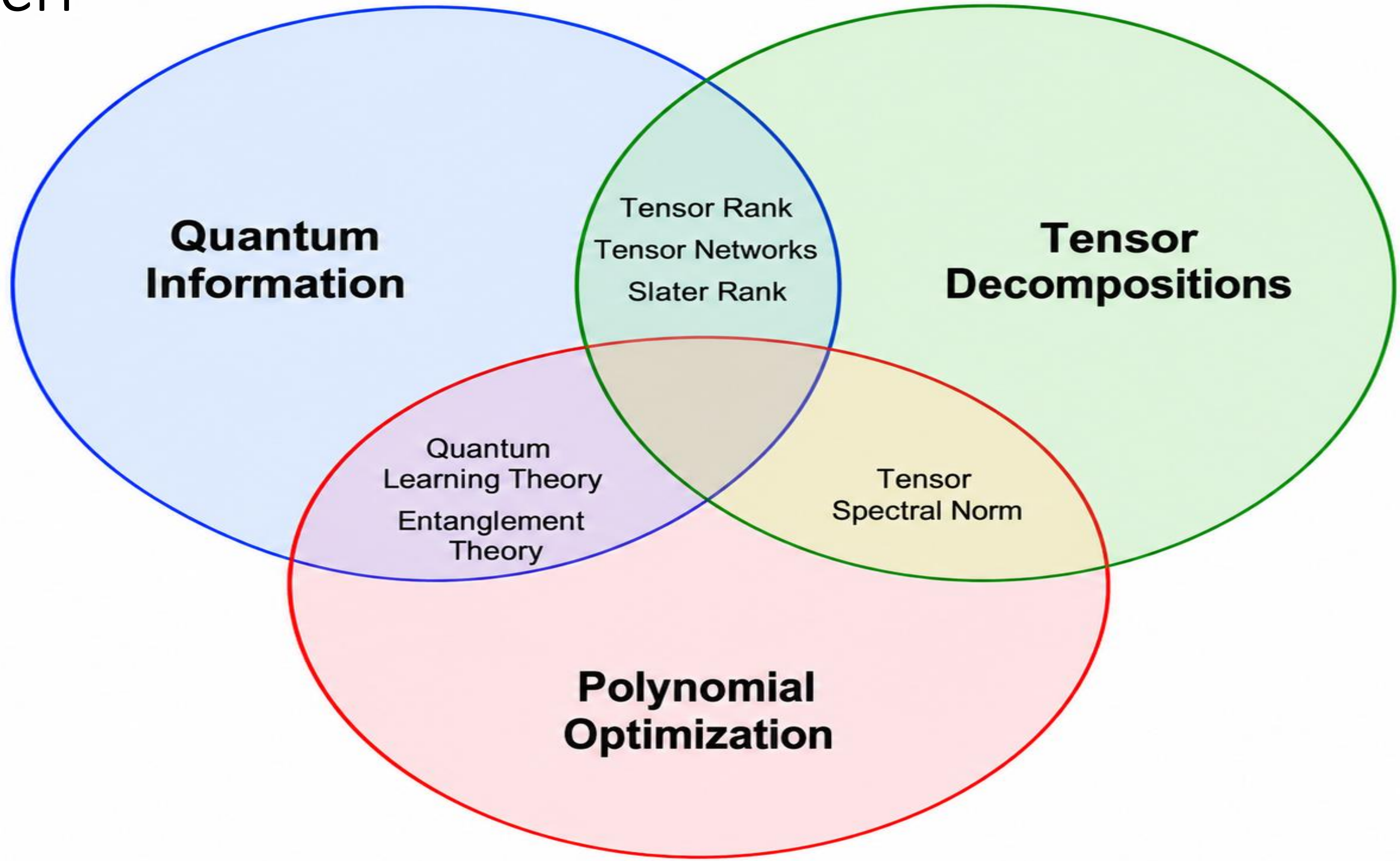
<u>OPEN QUESTION</u>: Suppose G is ϵ-far from the set of k-incoherent Gram matrices. How large does n need to be so that such a P_n does not exist?

<u>NOTE</u>: The Fact also holds when S_k is replaced by any affine cone variety X, and k-incoherence is replaced by the convex hull of $\{vv^*: v \in X\}$.

The only cases when I can answer this question is when X is a highest weight orbit

(e.g. $X = \mathrm{Veronese}$).

My research


Quantum Information
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Polynomial Optimization

I am hiring students and postdocs!

- CSSE dept, Concordia University
- Located in Montreal, Canada
- Please reach out if interested