Computing linear sections of varieties: quantum entanglement, tensor decompositions and beyond

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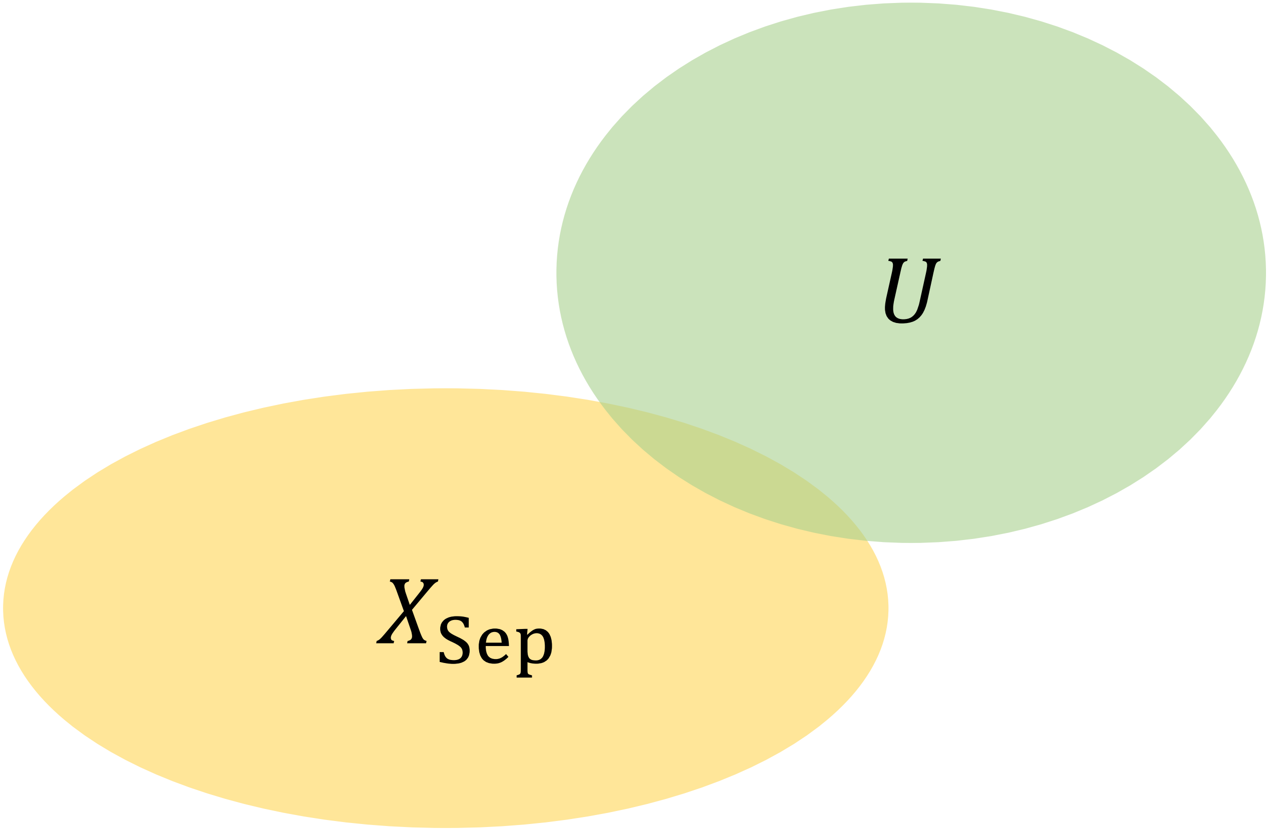
Product tensors: $X_{\text{Sep}} = \{u \otimes v : \; u, v \in \mathbb{C}^n\} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, determine if U is entangled, i.e. if $U \cap X_{\text{Sep}} = \{0\}$.



Applications: Quantum Information

- Range criterion: For a density operator $\rho \in D(\mathbb{C}^n \otimes \mathbb{C}^n)$,
  $\mathrm{Im}(\rho)$ entangled $\Rightarrow$ ρ entangled
- Entangled subspaces can be used to construct entanglement witnesses and quantum error-correcting codes

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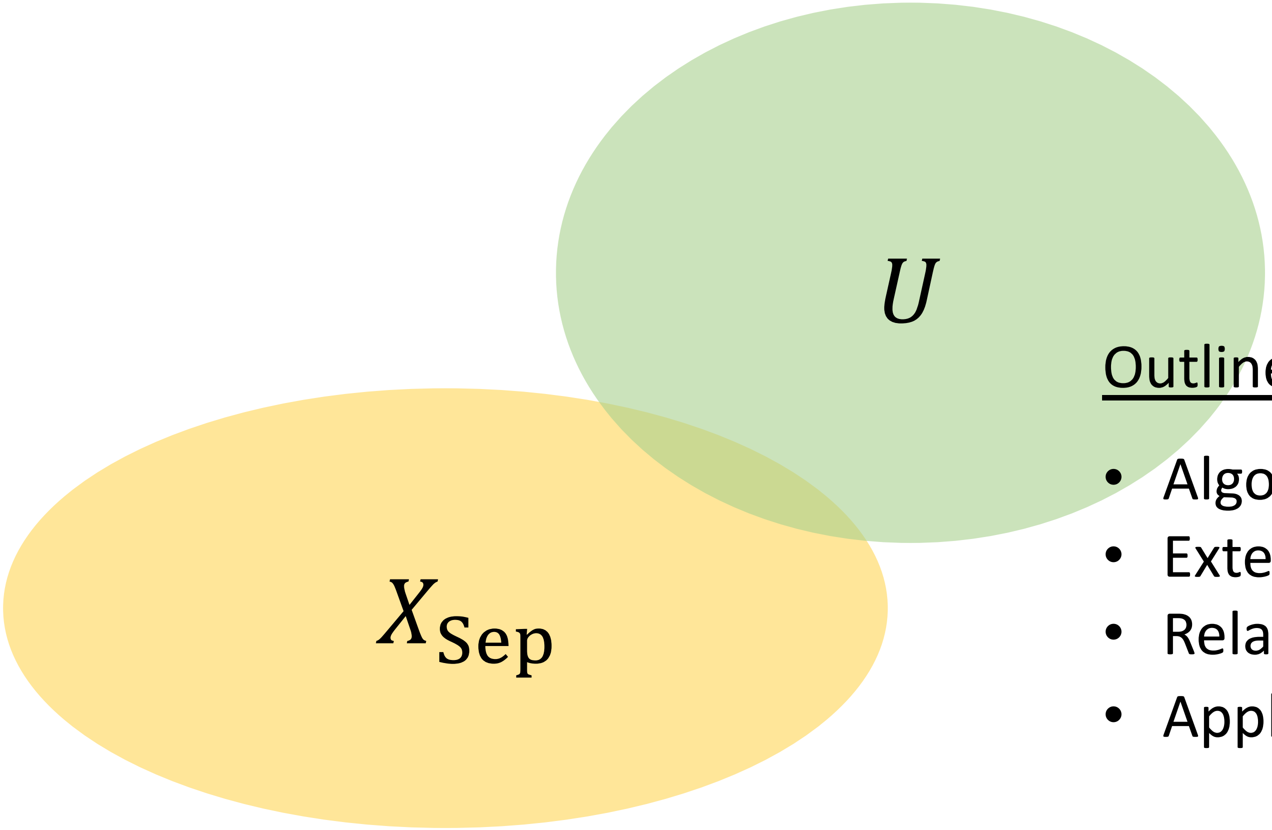

Outline:

- Algorithm + complete hierarchy (Nullstellensatz)
- Extension to other varieties X
- Related algorithm to recover elements of $U \cap X_{\text{Sep}}$
- Application to tensor decompositions

Product tensors: $X_{\mathrm{Sep}} = \{u \otimes v: \quad u, v \in \mathbb{C}^n\} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$,
determine if U is entangled, i.e. if $U \cap X_{\mathrm{Sep}} = \{0\}$.

[Barak et al 2019]: This is NP-Hard in the worst case.

[Barak et al 2019]: Best known algorithm takes $2^{\tilde{O}(\sqrt{n})}$ time.

[Classical AG, Parthasarathy 01]: $\dim(U) > (n-1)^2 \quad \Rightarrow \quad U$ is not entangled

U generic and $\dim(U) \leq (n-1)^2 \quad \Rightarrow \quad U$ is entangled

"Hay in a haystack problem"

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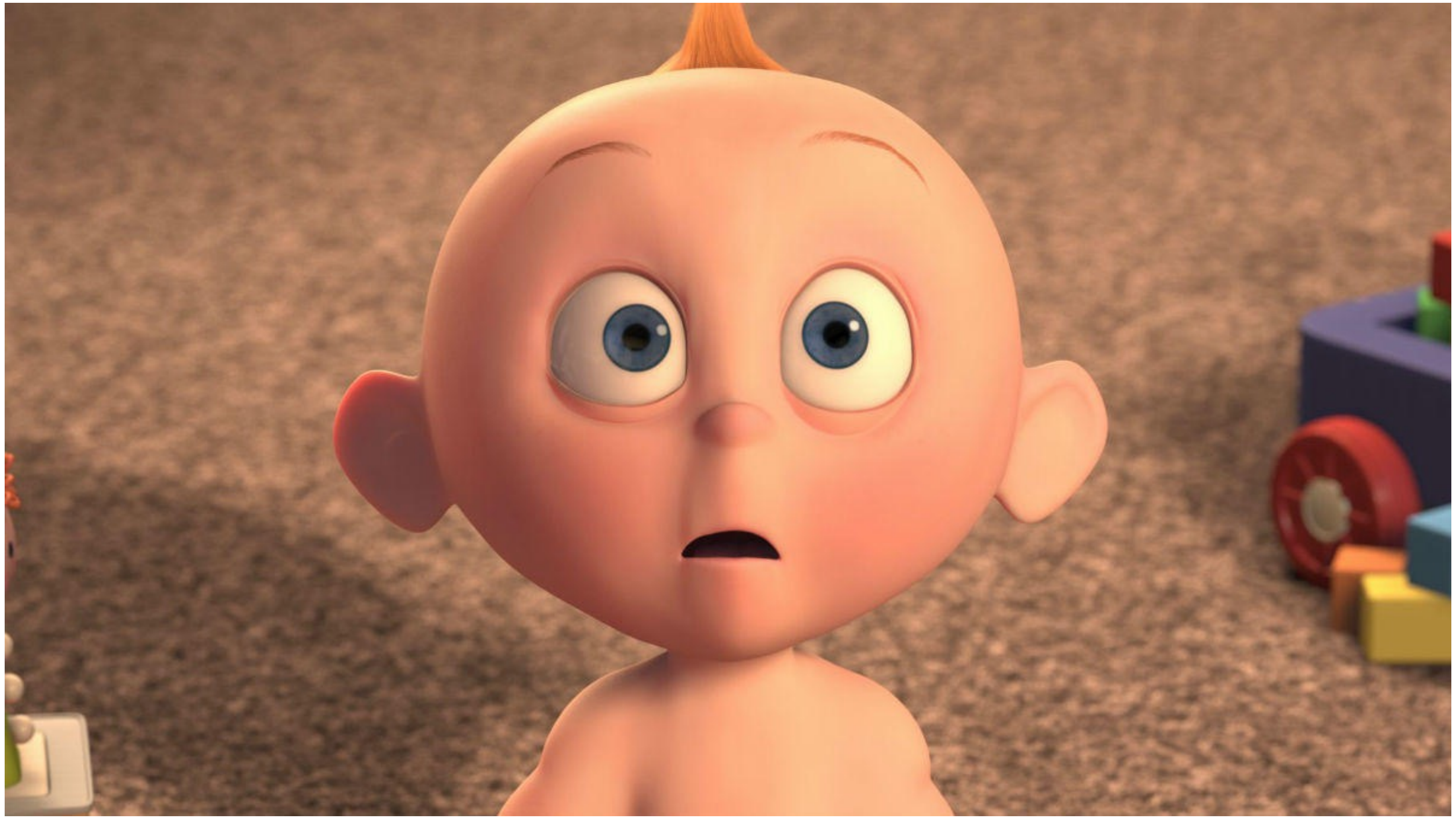
Algorithm for babies



Product tensors: $X_{\text{Sep}} = \{u \otimes v : \ u, v \in \mathbb{C}^n\} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, determine if U is entangled, i.e. if $U \cap X_{\text{Sep}} = \{0\}$.

Idea: Problem is difficult because it's non-linear

($X_{\text{Sep}} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$ isn't a linear subspace).

Make it linear: Instead check if $U \cap \text{Span}(X_{\text{Sep}}) = \{0\}$.

Doesn't work: $\text{Span}(X_{\text{Sep}}) = \mathbb{C}^n \otimes \mathbb{C}^n$.

Lift it up: Let $X_{Sep}^d = \text{Span}\{(u \otimes v)^{\otimes d} : \ u, v \in \mathbb{C}^n\} = S^d(\mathbb{C}^n) \otimes S^d(\mathbb{C}^n)$

Check if $U^{\otimes d} \cap X_{Sep}^d = \{0\}$.

Works extremely well already for $d = 2$!

Algorithm for adults

Product tensors: $X_{\text{Sep}} = \{u \otimes v: \quad u, v \in \mathbb{C}^n\} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, determine if U is entangled, i.e. if $U \cap X_{\text{Sep}} = \{0\}$.

Hilbert's Nullstellensatz:

$U \cap X = \{0\} \quad \Leftrightarrow \quad$ For some $d \in \mathbb{N}$ it holds that

$$I(U)_d + I(X_{Sep})_d = \mathbb{C}[x_{1,1}, \dots, x_{n,n}]_d$$

$$\Leftrightarrow$$

$$S^d(U) \cap I(X_{\text{Sep}})_d^{\perp} = \{0\}$$

$I(X_{\text{Sep}})_d^{\perp} = X_{Sep}^d$

$$\Leftrightarrow$$

$$U^{\otimes d} \cap X_{Sep}^d = \{0\}$$

Works extremely well already for $d = 2$!

Product tensors: $X_{\text{Sep}} = \{u \otimes v: \quad u, v \in \mathbb{C}^n\} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, determine if U is entangled, i.e. if $U \cap X_{\text{Sep}} = \{0\}$.

$$X_{Sep}^2 := \text{Span}\{(u \otimes v)^{\otimes 2}: \ u, v \in \mathbb{C}^n\} = S^2(\mathbb{C}^n) \otimes S^2(\mathbb{C}^n)$$

Takes poly(n) time to check

Algorithm:

If $U^{\otimes 2} \cap X_{\text{Sep}}^2 = \{0\}$, output U is entangled

Otherwise, output Fail

Correctness: $u \otimes v \in U \ \Rightarrow (u \otimes v)^{\otimes 2} \in U^{\otimes 2} \cap X_{\text{Sep}}^2$

$\Rightarrow$ Algorithm outputs Fail.

Product tensors: $X_{\mathrm{Sep}} = \{u \otimes v: \quad u, v \in \mathbb{C}^n\} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$,
determine if U is entangled, i.e. if $U \cap X_{\mathrm{Sep}} = \{0\}$

Works-Extremely-Well Theorem [JLV 22]:
U generic and $\dim(U) \leq \frac{1}{4}(n-1)^2 \quad \Rightarrow \quad U^{\otimes 2} \cap X_{Sep}^2 = \{0\}$.

Takes poly(n) time to check

Algorithm:
If $U^{\otimes 2} \cap X_{\mathrm{Sep}}^2 = \{0\}$, output U is entangled
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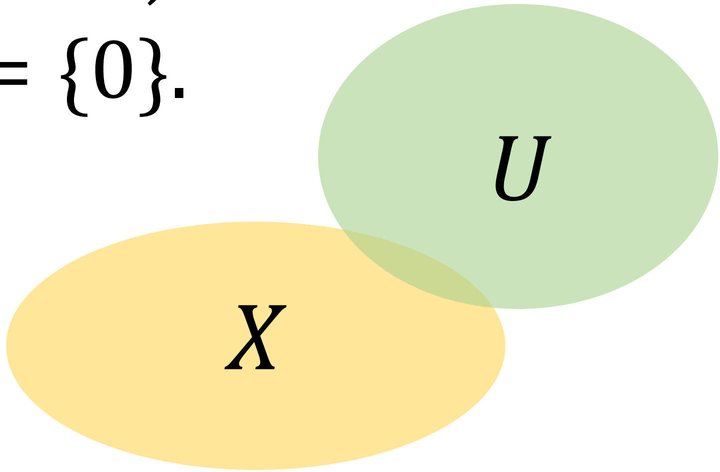
- Algorithm + complete hierarchy (Nullstellensatz)
- Extension to other varieties X
- Related algorithm to recover elements of $U \cap X_{\mathrm{Sep}}$
- Applications to tensor decompositions

Let $X \subseteq \mathbb{C}^N$ be a conic variety (for example, $X = X_{\text{Sep}} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$)

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^N$, determine if U avoids X, i.e. if $U \cap X = \{0\}$.



$$X^d := \text{Span}\{v^{\otimes d} : \ v \in X\}$$

Algorithm d:

Takes $N^{O(d)}$ time to check

If $U^{\otimes d} \cap X^d = \{0\}$, output U avoids X

Otherwise, output Fail

Completeness [Hilbert]: For $d = 2^{O(N)}$, Fail $\Leftrightarrow$ U intersects X

Derksen's proof (sketch)

*A slightly weaker WEW Theorem appears in [JLV 22] with a different proof.

WEW Theorem [Derksen]*: If $I \subseteq \mathbb{C}[x_1, \dots, x_N]$ is a homogeneous ideal and R is a non-negative integer such that

$$\dim I_d^\perp < \binom{N-R+d}{d},$$

then there exists an R-dimensional subspace $U \subseteq \mathbb{C}^D$ such that $U^{\otimes d} \cap I_d^\perp = \{0\}$.

Proof sketch: By a theorem of Galligo, after a linear change of coordinates wma $J := \mathrm{lm}(I)$ is Borel-fixed with respect to the reverse lexicographic monomial order.

If $x_R^d \notin J_d$, then $J_d \subseteq \langle x_1, \dots, x_{R-1} \rangle_d$. But then $\dim\left(I_d^\perp\right) = \dim\left(J_d^\perp\right)$

$$\geq \dim\left(\mathbb{C}[x_1, \dots, x_N]_d / \langle x_1, \dots, x_{R-1} \rangle_d\right)$$

$= \binom{N-R+d}{d}$, a contradiction.

So $x_R^d \in J_d$. But this implies all monomials in $x_1, \dots, x_R$ of degree d lie in J.

It follows that $S^d(U) \cap I_d^\perp = \{0\}$ for $U = \mathrm{span}\{e_1, \dots, e_R\}$. ∎

Examples

WEW Theorem [JLV 22]: For generic U of dimension $\dim(U) \leq$ 👽 it holds that $U^{\otimes d} \cap X^d = \{0\}$, for $d =$ 🐜

Schmidt rank $\leq r$ tensors

$X_r = \{v \in \mathbb{C}^n \otimes \mathbb{C}^n : \text{Schmidt−rank}(v) \leq r\}$

👽 $= \Omega_r(n^2)$
🐜 $= r + 1$

Product tensors in-X_{Sep}-arable ↔ Completely entangled

$X_{\text{Sep}} = \{v_1 \otimes \cdots \otimes v_m : v_i \in \mathbb{C}^n\}$

👽 $\sim(1/4)n^m$
🐜 $= 2$

Biseparable tensors in-X_B-arable ↔ Genuinely entangled

$X_B = \{v \in (\mathbb{C}^n)^{\otimes m}$: Some bipartition of v has rank 1$\}$

👽 $\sim(1/4)n^m$
🐜 $= 2$

Slice rank 1 tensors

$X_S = \{v \in (\mathbb{C}^n)^{\otimes m}$: Some 1 v.s. rest bipartition of v has rank 1$\}$

👽 $\sim(1/4)n^m$
🐜 $= 2$

Matrix product tensors of bond dimension $\leq r$

$X_{MPS} = \{v \in (\mathbb{C}^n)^{\otimes m}$: Every left-right bipartition has rank $\leq r\}$

👽 $= \Omega_r(n^m)$
🐜 $= r + 1$

Examples

WEW Theorem [JLV 22]: For generic U of dimension $\dim(U) \leq$ 👽 it holds that $U^{\otimes d} \cap X^d = \{0\}$, for $d =$ 🐜

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Product tensors in-X_{Sep}-arable ↔ Completely entangled

$X_{\text{Sep}} = \{v_1 \otimes \cdots \otimes v_m : v_i \in \mathbb{C}^n\}$

👽 $\sim(1/4)n^m$
🐜 $= 2$

Bisepara

$X_B = \{v$

n^m

Takeaway: Algorithm certifies entanglement of subspaces of dimension a constant multiple of the maximum possible in polynomial time.

Slice rank 1 tensors

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Matrix product tensors of bond dimension $\leq r$

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Suppose $U \subseteq \mathbb{C}^N$ has a basis $\{v_1, \dots, v_R\}$ such that each $v_i \in X$.

Problem: Given some other basis $\{u_1, \dots, u_R\}$ of U, recover $\{v_1, \dots, v_R\}$ (up to scale).

Example: Jennrich's Algorithm: If $U' \subseteq S^d(\mathbb{C}^N)$ is spanned by $\{v_1^{\otimes d}, \dots, v_R^{\otimes d}\}$ with $\{v_1, \dots, v_R\}$ linearly independent, then $\{v_1^{\otimes d}, \dots, v_R^{\otimes d}\}$ can be recovered from any basis of U' in $N^{O(d)}$- time.

Jennrich's Algorithm:

Pick $T_j \in U'$, $j = 1,2$ at random, view these as maps $T_j : (\mathbb{C}^N)^{\otimes d-1} \to \mathbb{C}^N$

$$T_j = \sum_{i=1}^R \alpha_{j,i} v_i \left(v_i^t\right)^{\otimes d-1} \qquad T_j^{-1} = \sum_i \frac{1}{\alpha_{j,i}} (w_i)^{\otimes d-1} w_i^t \qquad \text{where } w_i^t v_{i'} = \delta_{i,i'}$$

So $T_1 T_2^{-1} = \sum_i \frac{\alpha_{1,i}}{\alpha_{2,i}} v_i w_i^t$. E-vectors / E-values of $T_1 T_2^{-1}$ are v_i , $\frac{\alpha_{1,i}}{\alpha_{2,i}}$ ← Distinct for different i

Suppose $U \subseteq \mathbb{C}^N$ has a basis $\{v_1, \dots, v_R\}$ such that each $v_i \in X$.

Works-Extremely-Well Theorem [JLV 22]:
If $d \geq 2$, X is irreducible, cut out in degree d, and has no equations in degree $d-1$, then (1) and (2) hold for generic $v_1, \dots, v_R \in X$ as long as $R \leq \frac{\dim(I(X)_d)}{d!\binom{N+d-1}{d}}(N+d-1)$

Pro ... le).

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Lifted Jennrich's Algorithm [JLV 2022]: Run Jennrich on $U' = U^{\otimes d} \cap X^d$.

$$v^{\otimes d} \in U' \iff v \in U \cap X$$

Generalizes FOOBI algorithm [DLCC '07]

For this to work, need:

1. $\{v_1^{\otimes d}, \dots, v_R^{\otimes d}\}$ spans U'.
2. $\{v_1, \dots, v_R\}$ is linearly independent.

Suppose $U \subseteq \mathbb{C}^N$ has a basis $\{v_1, \dots, v_R\}$ such that each $v_i \in X$.

Pro

le).

Exa

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Works-Extremely-Well Theorem [JLV 22]:

If $d \geq 2$, X is irreducible, cut out in degree d, and has no equations in degree $d-1$, then (1) and (2) hold for generic $v_1, \dots, v_R \in X$ as long as $R \leq \frac{\dim(I(X)_d)}{d!\binom{N+d-1}{d}}(N+d-1)$

Proof technique: Show that

$$I(X)_d + I(U)_d = I(v_1, \dots, v_R)_d$$

for generic $v_1, \dots, v_R \in X$. This is equivalent to (1).

Lifted Jennrich's Algorithm [JLV 2022]: Run Jennrich on $U' = U^{\otimes d} \cap X^d$.

Compare with earlier result:

$I(X)_d + I(U)_d = S^d(\mathbb{C}^N)$ for generic $v_1, \dots, v_R \in \mathbb{C}^N$

For this to work, need:

1. $\{v_1^{\otimes d}, \dots, v_R^{\otimes d}\}$ spans U'.
2. $\{v_1, \dots, v_R\}$ is linearly independent.

Q: Clean algebraic proof?

Similar WEW Theorems were claimed in [DL 06, DLCC 07] for the special case $X = X_{\text{Sep}}$, but their proofs are incorrect.

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Application: (X, k)-decompositions

For $T \in V \otimes \mathbb{C}^k$, an (X, k)-decomposition is an expression

$$T = \sum_{i=1}^{R} v_i \otimes z_i \in V \otimes \mathbb{C}^k$$

where $v_1, \ldots, v_R \in X$

<u>Example:</u> When $X = X_{\text{Sep}} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, an (X, k)-decomposition is just a tensor decomposition.

Viewing T as a map $\mathbb{C}^k \to V$, each $v_i \in T(\mathbb{C}^k) \cap X$,

so computing $T(\mathbb{C}^k) \cap X \leftrightarrow (X, k)$-decomposing T

(Assuming that $\{z_1, \ldots, z_R\}$ is linearly independent)

Corollary to WEW Theorem [JLV 22]: A generic tensor $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^k$ with

$$\operatorname{rank}(T) \leq \min\{\frac{1}{4}(n-1)^2, k\}$$

has a unique rank decomposition, which is recovered in POLY(n)-time by applying our algorithm to $T(\mathbb{C}^k)$.

In particular, a generic $n \times n \times n^2$ tensor of rank $\sim \frac{1}{4}n^2$ is recovered by algorithm.

Corollary to WEW Theorem [JLV 22]: A generic tensor $T \in (\mathbb{C}^n)^{\otimes m}$ of tensor rank

$$\mathrm{rank}(T) = O(n^{\lfloor m/2 \rfloor})$$

has a unique tensor rank decomposition, which is recovered in $n^{O(m)}$-time by applying our algorithm to $T\left((\mathbb{C}^n)^{\otimes \lfloor m/2 \rfloor}\right)$.

(This is new when m is even. When m is odd you can just use Jennrich directly.)

Corollary to WEW Theorem [JLV 22]: A generic tensor $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^k$ of r-aided rank

$$\mathrm{r-aided\ rank}(X) \leq \min\{\Omega_r(n^2), k\}$$

has a unique tensor rank decomposition, which is recovered in $n^{O(r)}$-time by applying our algorithm to $T(\mathbb{C}^k)$.

$T = \sum_i v_i \otimes w_i$, where $v_i \in X_r$

r-aided rank $\iff$ $(r, r, 1)$-multilinear rank

Conclusion

1. **Complete hierarchies** of linear systems to certify entanglement of a subspace. These work extremely well already at early levels.

Title: Complete hierarchy of linear systems for certifying quantum entanglement of subspaces

2. **(Briefly mentioned) poly-time algorithms** to find low-entanglement elements of a subspace. These also work extremely well.

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3. **Extending symmetric extensions:** Separability testing hierarchy of [DPS 04] extended to hierarchies for Schmidt number, biseparability, and X-arability.

Title: TBD

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