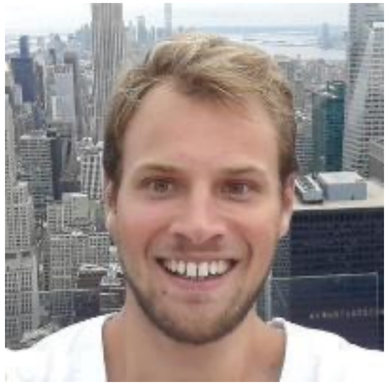
New techniques for bounding stabilizer rank



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Random tensors at CIRM 2022

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Computational basis

- Let $\{|0\rangle, |1\rangle\} \subseteq \mathbb{C}^2$ be the computational basis for $\mathbb{C}^2$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Let $\{|x\rangle : x \in \mathbb{F}_2^n\} \subseteq (\mathbb{C}^2)^{\otimes n}$ be the computational basis for $(\mathbb{C}^2)^{\otimes n}$

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- A state is a unit vector in $(\mathbb{C}^2)^{\otimes n}$ (mod phase, i.e. an element of $\mathbb{P}^{2^n-1}$).
- We often omit normalization.
- States in $\mathbb{C}^2$ are called qubits.
- States are denoted $|\psi\rangle, |\phi\rangle$, etc.
- $\langle\psi|$ denotes conjugate-transpose of $|\psi\rangle$

Quantum circuits

General framework:

1. Prepare a computational basis state $|0\cdots 0\rangle \in (\mathbb{C}^2)^{\otimes n}$.
2. Apply a unitary matrix $U|0\cdots 0\rangle$
3. Measure in the computational basis. For $x \in \mathbb{F}_2^n$, $p(x) = |\langle x|U|0\cdots 0\rangle|^2$.

Quantum circuits

General framework:

1. Prepare a computational basis state $|0\cdots 0\rangle \quad \in (\mathbb{C}^2)^{\otimes n}$.
2. Apply a unitary matrix $U|0\cdots 0\rangle$

$k \leq n$

3. Measure in the computational basis. For $x \in \mathbb{F}_2^k$, $p(x) = ||(\langle x| \otimes \mathbb{I})U|0\cdots 0\rangle\,||^2$

Quantum circuits

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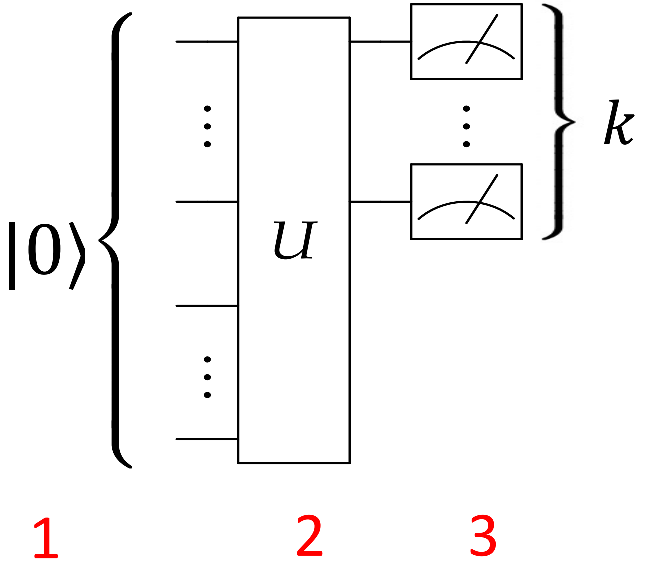

This talk: Classical simulation of Clifford+T circuits via stabilizer rank

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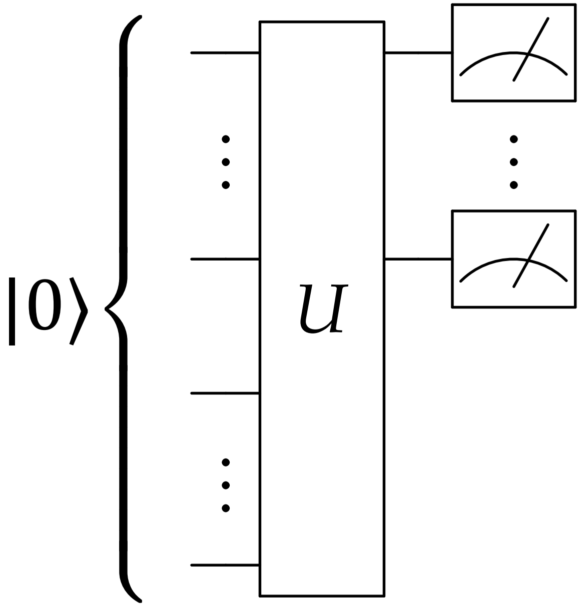
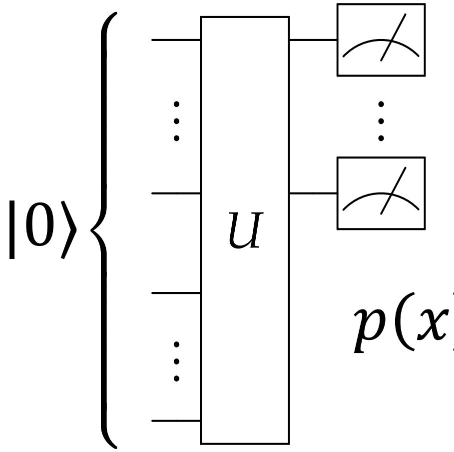
Classical simulation of quantum circuits

<u>Question:</u> Given a classical description of a quantum circuit



Can it be simulated efficiently on a classical computer?

Types of simulation



$x \in \mathbb{F}_2^k$

$p(x) = ||(\langle x| \otimes \mathbb{I})U|0\rangle^{\otimes n}||^2$

- Strong simulation:

  Compute $p(x)$ for all $x \in \mathbb{F}_2^k$.

- ϵ-Strong simulation:

  Find a probability vector $\tilde{p}$ such that

  $$(1-\epsilon)p(x) \leq \tilde{p}(x) \leq (1+\epsilon)p(x) \quad \text{for all } x \in \mathbb{F}_2^k.$$

- Weak simulation:

  Sample elements of $x \in \mathbb{F}_2^k$ from a probability distribution $\tilde{p}$ such that

  $$||\tilde{p} - p||_1 \leq \epsilon$$

This talk: Classical simulation of Clifford+T circuits via stabilizer rank

Clifford circuits

The Clifford group is the group of unitaries $U: (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$ composed of Clifford gates

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

[Gottesman-Knill 98]: Clifford circuits can be efficiently simulated.

Clifford+T circuits

The Clifford+T group is the unitary group $U: (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$ composed of Clifford gates

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

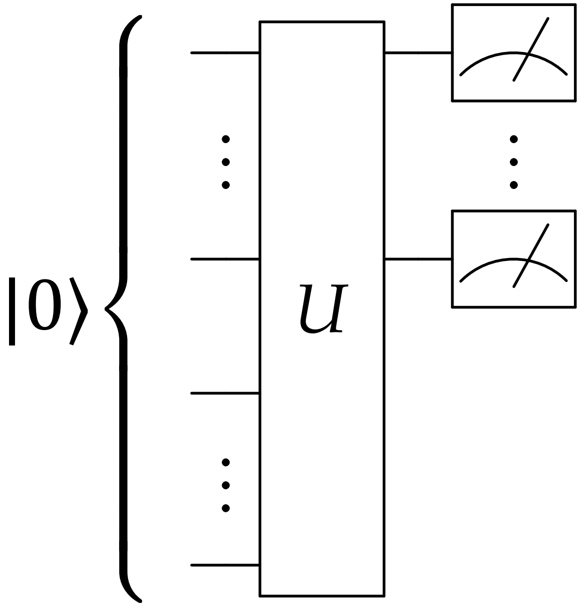
and T-gates

$$\mathrm{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}.$$

<u>Motivation [Bravyi-Kitaev 05]:</u> The Clifford+T group is "universal" for quantum computation … i.e. its closure is all unitaries.

Classical simulation of Clifford+T circuits

<u>Question:</u> Given a classical description of a Clifford+T circuit



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Classical simulation of Clifford+T circuits

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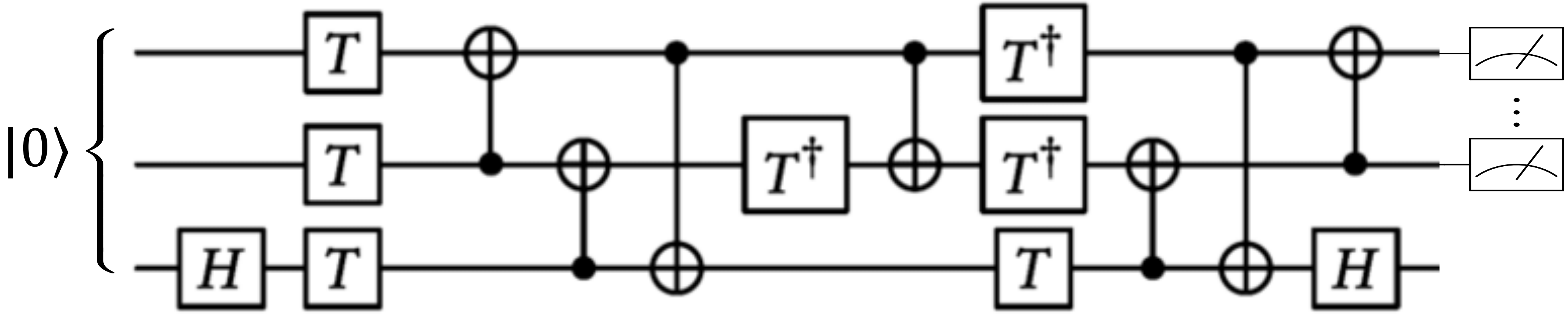


Can it be simulated efficiently on a classical computer?

This talk: Classical simulation of Clifford+T circuits via stabilizer rank

Partial answer: Stabilizer rank

- A state $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ is a stabilizer state if $|\phi\rangle = U|0\rangle^{\otimes n}$ for some Clifford circuit U.

- The stabilizer rank of a state $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$, denoted $\chi(|\psi\rangle)$, is the minimum number r for which

$$|\psi\rangle = \sum_{i=1}^{r} c_i |\phi_i\rangle$$

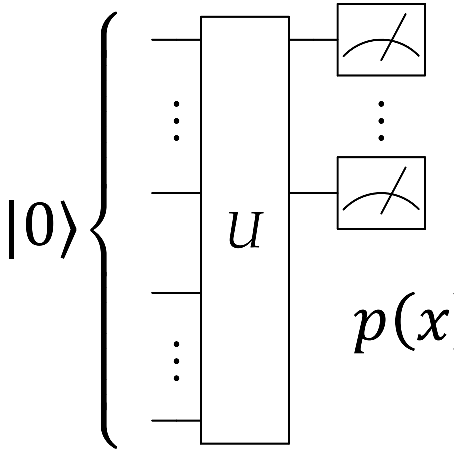
  for some $c_i \in \mathbb{C}$ and $|\phi_i\rangle$ stabilizer states.

- The δ-approximate stabilizer rank of $|\psi\rangle$ is

$$\chi_\delta(|\psi\rangle) = \min\{\chi(|\mu\rangle) : \| |\psi\rangle - |\mu\rangle \| \le \delta\}.$$

Partial answer: Stabilizer rank $|H\rangle = |0\rangle + (1+\sqrt{2})|1\rangle \approx |0\rangle + 2.41|1\rangle$

[Bravyi-Smith-Smolin 16, Bravyi-Gosset 16]: A Clifford+T circuit with n T-gates can be simulated…



$x \in \mathbb{F}_2^k$

$$p(x) = ||(\langle x| \otimes \mathbb{I})U|0\rangle^{\otimes n}||^2$$

- Strongly with cost quadratic in $\chi\left(|H\rangle^{\otimes n}\right)$.

  Compute $p(x)$ for all $x \in \mathbb{F}_2^k$

- ϵ-Strongly with cost linear in $\chi\left(|H\rangle^{\otimes n}\right)$.

  Find a probability vector $\tilde{p}$ such that $(1-\epsilon)p(x) \le \tilde{p}(x) \le (1+\epsilon)p(x)$

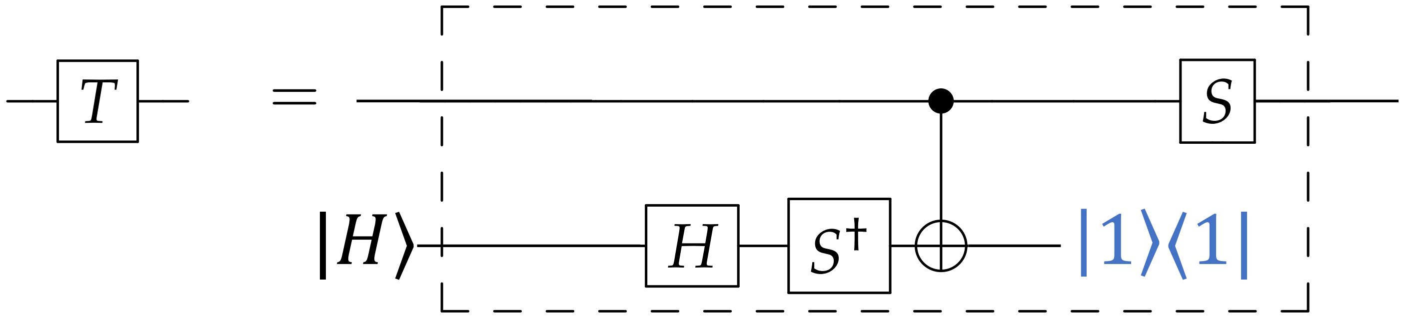
- Weakly with cost linear in $\chi_\delta\left(|H\rangle^{\otimes n}\right)$.

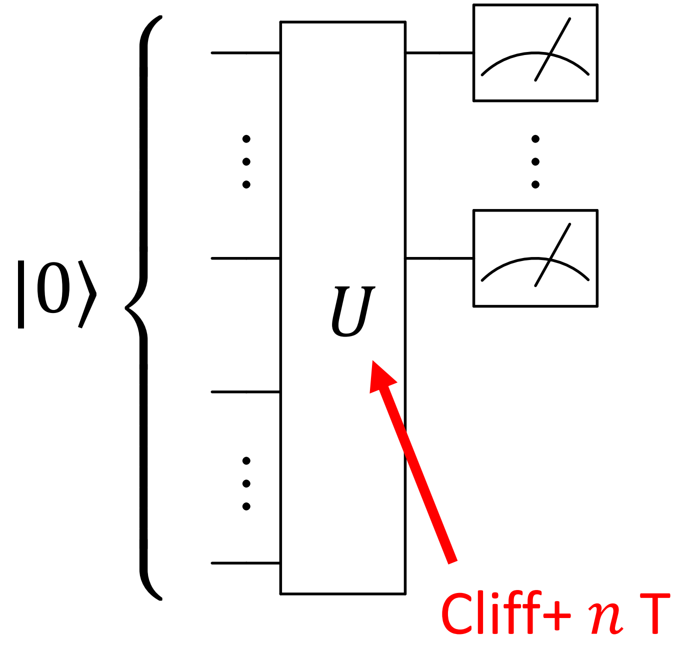
  Sample elements of $x \in \mathbb{F}_2^k$ from a probability distribution $\tilde{p}$ such that

$$\|\tilde{p} - p\|_1 \le \epsilon$$

Proof idea

[GK98]: Clifford circuits can be efficiently simulated.





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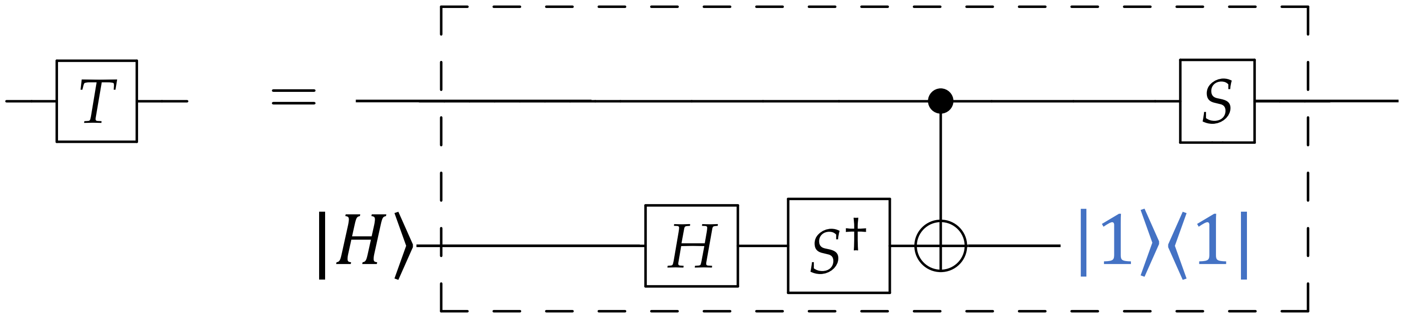
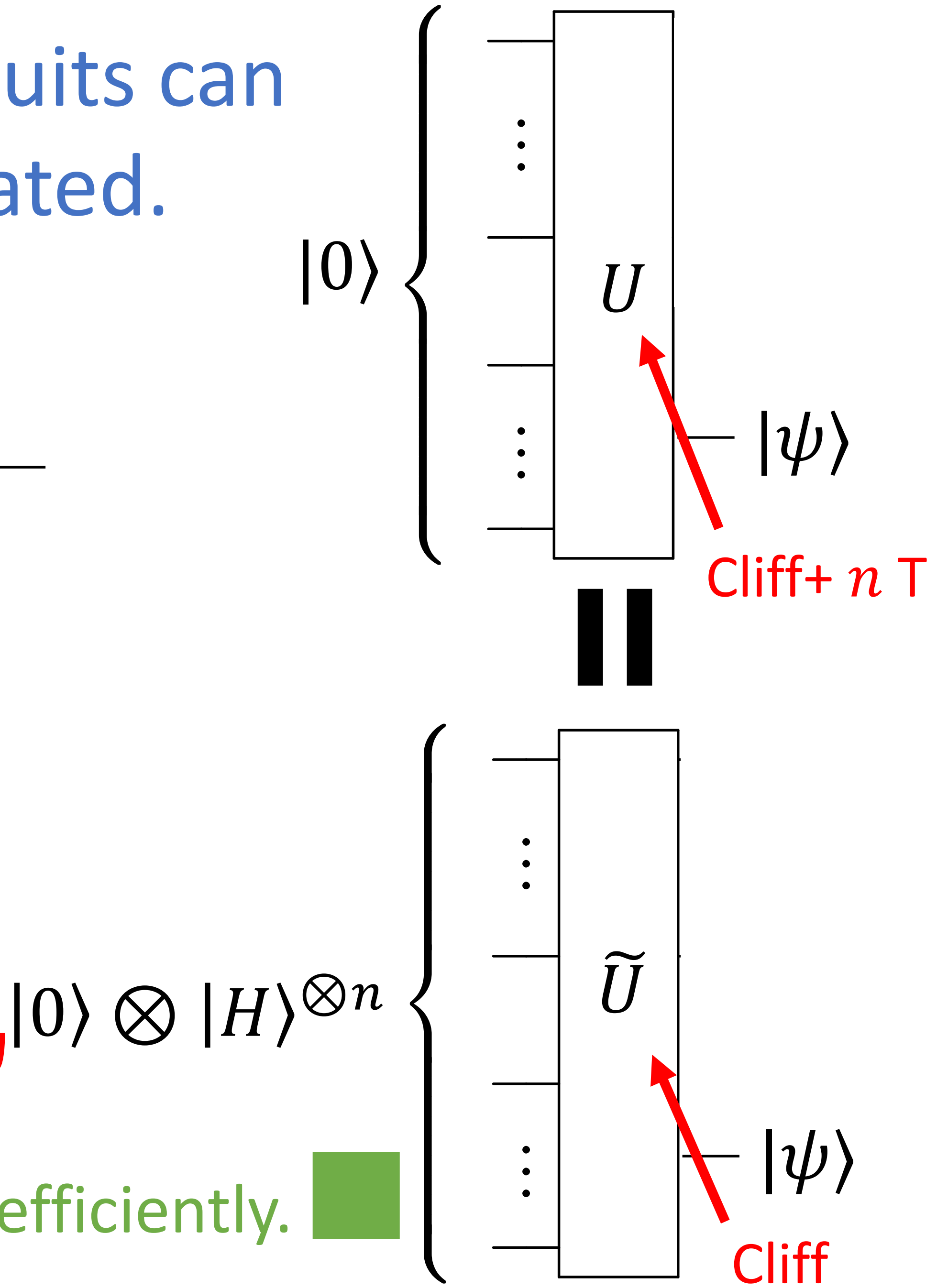




Let $r = \chi\left(|H\rangle^{\otimes n}\right)$ and $|H\rangle^{\otimes n} = \sum_{i=1}^{r} c_i |\phi_i\rangle$.

$$|\psi\rangle = \widetilde{U}(|0\rangle \otimes |H\rangle^{\otimes n}) = \sum_{i=1}^{r} c_i \underbrace{\widetilde{U}(|0\rangle \otimes |\phi_i\rangle)}_{=\widetilde{\widetilde{U}}_i|0\rangle}$$

By [GK98], can simulate measurements on each efficiently. ■

Known bounds on stabilizer rank

$\chi\left(|H\rangle^{\otimes n}\right)$

- [Huang-Newman-Szegedy 20]: $\chi\left(|H\rangle^{\otimes n}\right)$ super-polynomial unless P=NP.
- [Bravyi-Smith-Smolin 16]: $\chi\left(|H\rangle^{\otimes n}\right) \geq \Omega(\sqrt{n})$.
- [Peleg-Shpilka-Volk 21]: $\chi\left(|H\rangle^{\otimes n}\right) \geq \Omega(n)$.
- [Qassim-Pashayan-Gosset 21]: $\chi\left(|H\rangle^{\otimes n}\right) \leq 2^{\alpha n}$, where $\alpha = \frac{1}{4}\log_2(3)$.

$\chi_\delta\left(|H\rangle^{\otimes n}\right)$

This talk: Alternate proofs up to log factor

- [Peleg-Shpilka-Volk 21]:

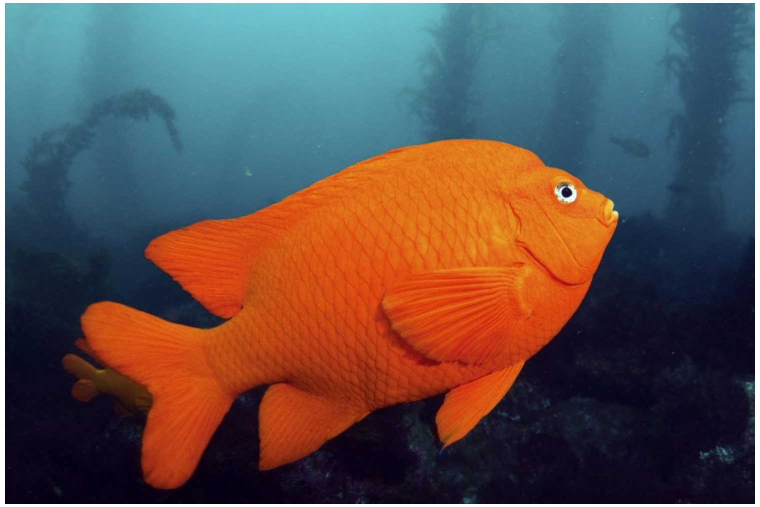
  There exists $\delta > 0$ such that $\chi_\delta\left(|H\rangle^{\otimes n}\right) \geq \Omega(\sqrt{n}/\log n)$
- [Bravyi-Gosset 16]: $\chi_\delta\left(|H\rangle^{\otimes n}\right) \leq O\left(\frac{1}{\delta^2} 2^{\alpha m}\right)$, where $\alpha \approx 0.228$.

Rest of talk

Lower bounds

Match [Peleg, Shpilka, Volk 22] up to log factor

- $\chi\left(|H\rangle^{\otimes n}\right) \geq \Omega(n/\log n)$.
- There exists $\delta > 0$ such that $\chi_\delta\left(|H\rangle^{\otimes n}\right) \geq \Omega(\sqrt{n}/\log n)$.



Upper bounds

- Generic stabilizer rank

Image: https://fishingbooker.com/blog/california-state-fish-golden-trout-garibaldi/

Fact: If $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ is a stabilizer state, then the coordinates of $|\phi\rangle$ are $\{0, \pm 1, \pm i\}$ (up to normalization).

Theorem [Dehaene, De Moor 03]:

$|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ is a stabilizer state $\Leftrightarrow$ $|\phi\rangle = \sum_{x \in A} i^{l(x)}(-1)^{q(x)}|x\rangle$,

where $A \subseteq \mathbb{F}_2^n$ is an affine linear subspace

$l\colon \mathbb{F}_2^n \to \mathbb{F}_2$ is a linear function

$q\colon \mathbb{F}_2^n \to \mathbb{F}_2$ is a quadratic polynomial

Subset-sum representations

- Let $\alpha \in \mathbb{C}^k, \beta \in \mathbb{C}^r$. We say β is a subset-sum representation of α if each α_j is equal to the sum of some subset of $\{\beta_1, \ldots, \beta_r\}$.

- Example: $\beta = (1,2)$ is a subset-sum representation of $\alpha = (1,2,3)$.

- Example: If $|\psi\rangle = \sum_{i=1}^{r} c_i |\phi_i\rangle$ is a stabilizer decomposition, then

$$\beta = (c_1, \ldots, c_r, -c_1, \ldots, -c_r, ic_1, \ldots, ic_r, -ic_1, \ldots, -ic_r) \in \mathbb{C}^{4r}$$

is a subset-sum representation of $|\psi\rangle$.

$|\phi_i\rangle$ stabilizer $\Rightarrow$coordinates are in $\{0, \pm 1, \pm i\}$

$\Rightarrow \chi(|\psi\rangle) \geq \frac{1}{4} \cdot$ (the size of the smallest subset-sum rep of $|\psi\rangle$)

Lower bounds on the size of a subset-sum rep

- Let $\alpha \in \mathbb{C}^k, \beta \in \mathbb{C}^r$. We say β is a subset-sum representation of α if each α_j is equal to the sum of some subset of $\{\beta_1, \dots, \beta_r\}$.

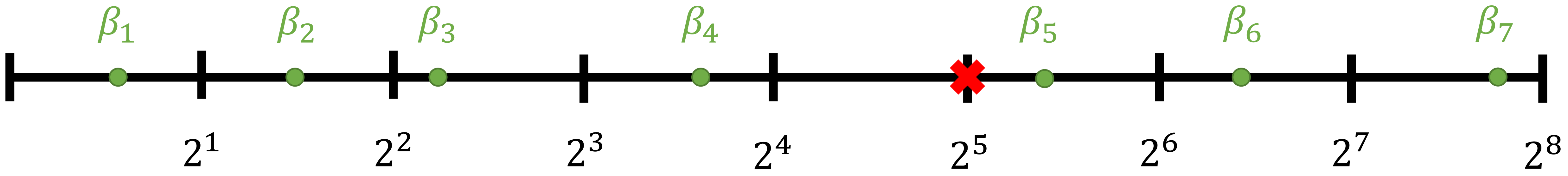
- Trivially, $\mathrm{r} \geq \log_2 k$, since $\{\beta_1, \dots, \beta_r\}$ has just 2^r different subsets.

α exponentially increasing

- Theorem [Moulton 01]: If $2|\alpha_j| \leq |\alpha_{j+1}|$ for all $j \in \{1, \dots, k-1\}$, then $r \geq k/\log_2 k$.

Linear in k, instead of logarithmic!

- Example: If $\alpha = (2^1, 2^2, \dots, 2^k)$, then $r \geq k/\log_2 k$



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- Theorem [Lovitz-Steffan]: If the coordinates of $|\psi\rangle$ contain an exponentially increasing sequence of length k, then $\chi(|\psi\rangle) \geq k/(4\log_2 k)$.

Lower bound on stabilizer rank

- <u>Theorem [Lovitz-Steffan]:</u> If the coordinates of $|\psi\rangle$ contain an exponentially increasing sequence of length k, then $\chi(|\psi\rangle) \geq k/(4\log_2 k)$.

<u>Corollary [Lovitz-Steffan]:</u> $\chi(|H\rangle^{\otimes n}) \geq n/(4\log_2 n)$.

Proof: Since $|H\rangle \approx |0\rangle + 2.41|1\rangle$,

$$|H\rangle^{\otimes n} \approx |0\cdots 0\rangle + (2.41)(|0\cdots 01\rangle + \cdots + |10\cdots 0\rangle) + \cdots + (2.41)^n|1\cdots 1\rangle.$$

$\Rightarrow |H\rangle^{\otimes n}$ contains the exponentially increasing sequence $(2.41, 2.41^2, \ldots, 2.41^n)$

$\Rightarrow \chi(|H\rangle^{\otimes n}) \geq n/(4\log_2 n)$ by boxed theorem.

$\blacksquare$

Lower bound on approximate stabilizer rank

- The δ-approximate stabilizer rank of a normalized state $|\psi\rangle$ is
$$\chi_\delta(|\psi\rangle) = \min\{\chi(|\mu\rangle) : \| |\psi\rangle - |\mu\rangle \| \le \delta\}.$$

- Theorem [Lovitz-Steffan]: There exists $\delta > 0$ for which $\chi_\delta(|H\rangle^{\otimes n}) \ge \sqrt{n}/(4\log_2 \sqrt{n})$.

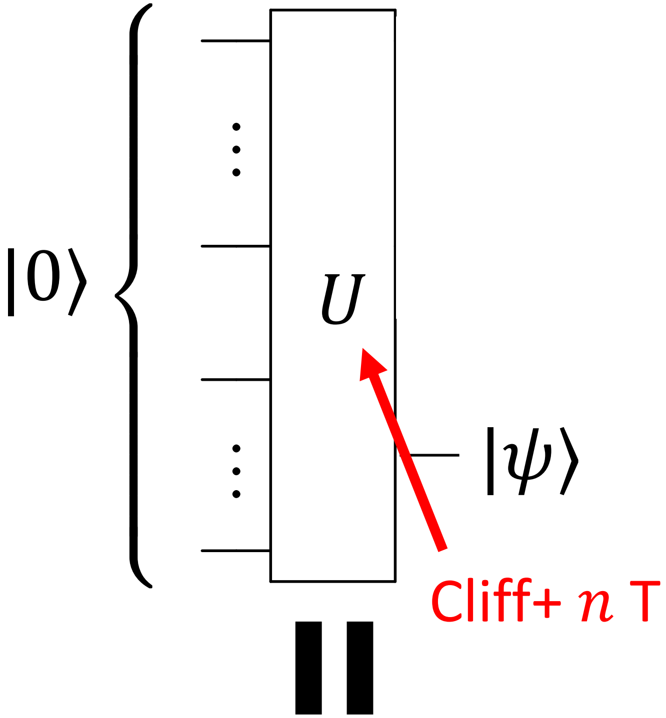
Proof sketch: Show that for δ small enough, any state that is δ-close to $|H\rangle^{\otimes n}$ must contain an exponentially increasing sequence of length $\sqrt{n}$ (Use De Moivre-Laplace).

Result follows from boxed theorem. $\blacksquare$

Super-linear lower bound on $\chi\left(|H\rangle^{\otimes n}\right)$?

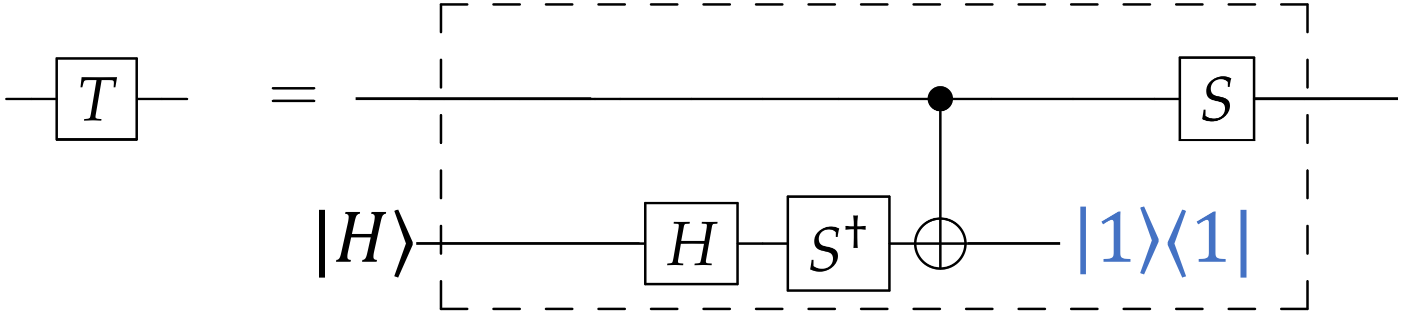
[BSS16] idea:

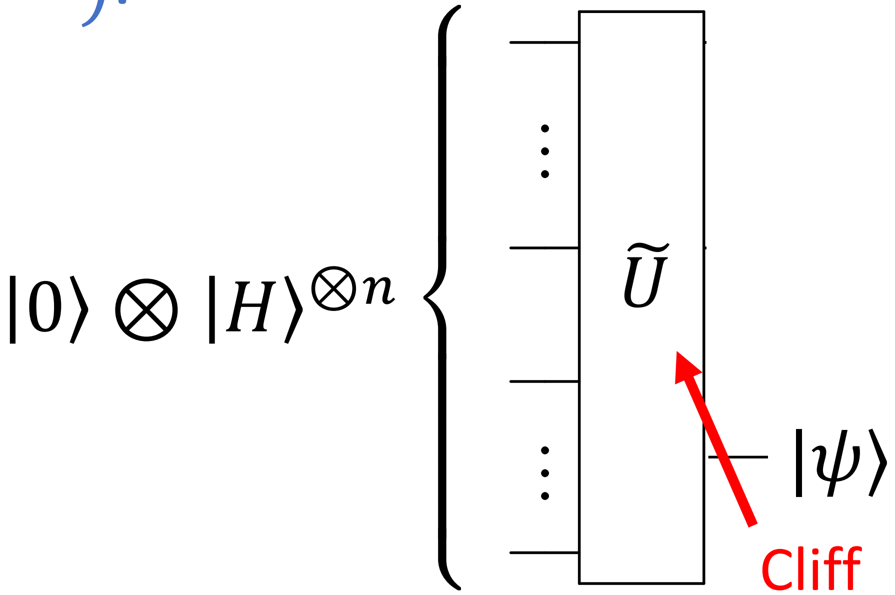
The T-count of a state $|\psi\rangle$ is the minimum number n of T gates needed to prepare $|\psi\rangle$ with a Cliff+T circuit U



Fact: If $|\psi\rangle$ has T-count n, then $\chi(|\psi\rangle) \leq \chi\left(|H\rangle^{\otimes n}\right)$.

Proof:





Super-linear lower bound on $\chi\left(|H\rangle^{\otimes n}\right)$?

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The T-count of a state $|\psi\rangle$ is the minimum number n of T gates needed to prepare $|\psi\rangle$ with a Cliff+T circuit U

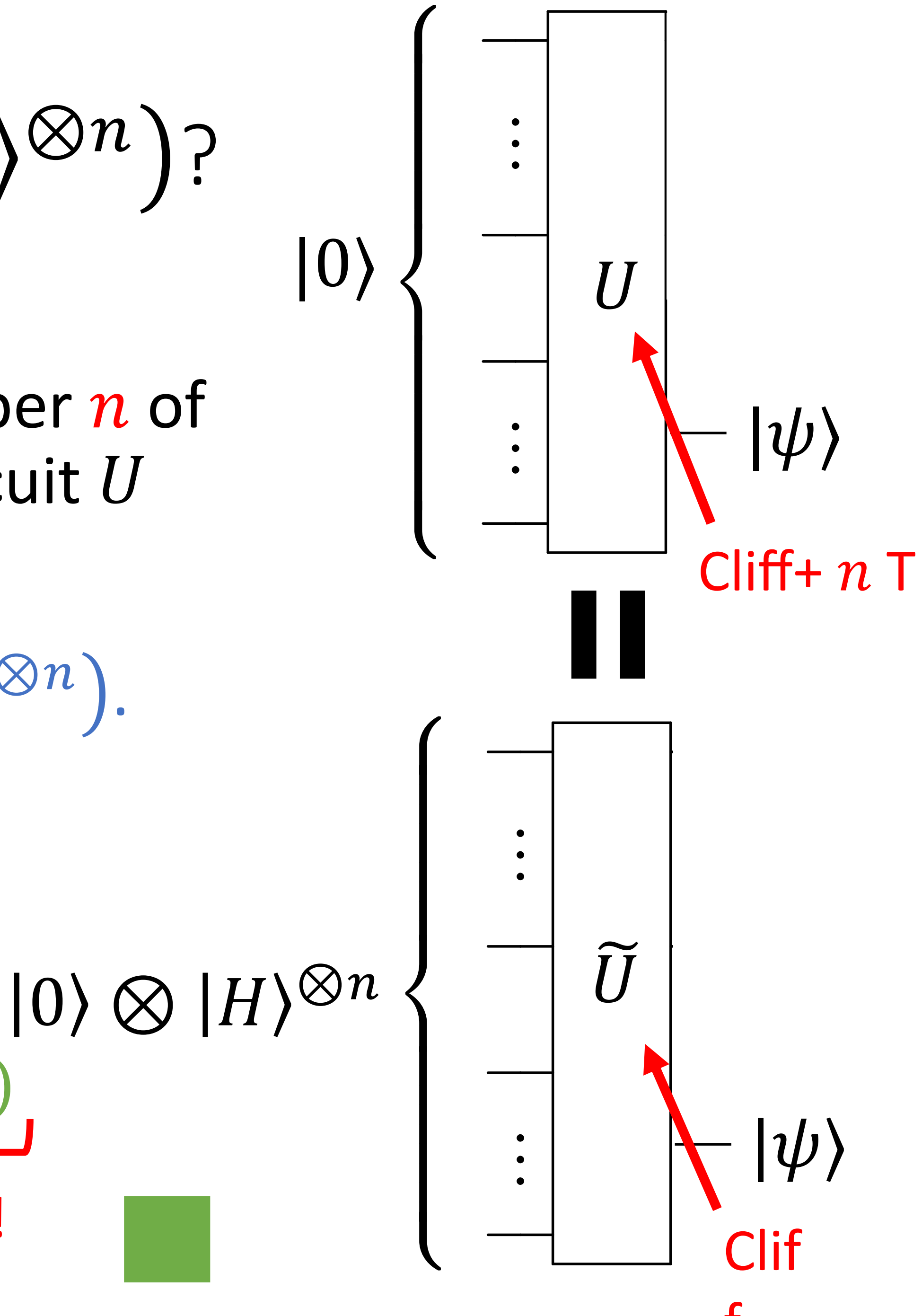
<u>Fact:</u> If $|\psi\rangle$ has T-count n, then $\chi(|\psi\rangle) \leq \chi\left(|H\rangle^{\otimes n}\right)$.

Proof:

Let $r = \chi\left(|H\rangle^{\otimes n}\right)$ and $|H\rangle^{\otimes n} = \sum_{i=1}^{r} c_i|\phi_i\rangle$.

$|\psi\rangle = \widetilde{U}(|0\rangle \otimes |H\rangle^{\otimes n}) = \sum_{i=1}^{r} c_i \underbrace{\widetilde{U}(|0\rangle \otimes |\phi_i\rangle)}_{\text{Stabilizer state!}}$

… so $\chi(|\psi\rangle) \leq r$.



Super-linear lower bound on $\chi\left(|H\rangle^{\otimes n}\right)$?

Fact: If $|\psi\rangle$ has T-count n, then $\chi(|\psi\rangle) \leq \chi\left(|H\rangle^{\otimes n}\right)$.

[BSS16] idea: For each n, if there exists a state $|\psi_n\rangle$ that:

1. Has T-count n
2. Satisfies $\chi(|\psi_n\rangle) \geq n^{1+\epsilon}$

… then $\chi\left(|H\rangle^{\otimes n}\right) \underset{1}{\geq} \chi(|\psi_n\rangle) \underset{2}{\geq} n^{1+\epsilon}$ ← Super-linear

Super-linear lower bound on $\chi\left(|H\rangle^{\otimes n}\right)$?

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[BSS16] idea: For each n, if there exists a state $|\psi_n\rangle$ that:

1. Has T-count n
2. Every subset-sum rep of $|\psi_n\rangle$ has size at least $n^{1+\epsilon}$

… then $\chi\left(|H\rangle^{\otimes n}\right) \underset{1}{\geq} \chi(|\psi_n\rangle) \underset{2}{\geq} \frac{1}{4} n^{1+\epsilon}$ ← Super-linear

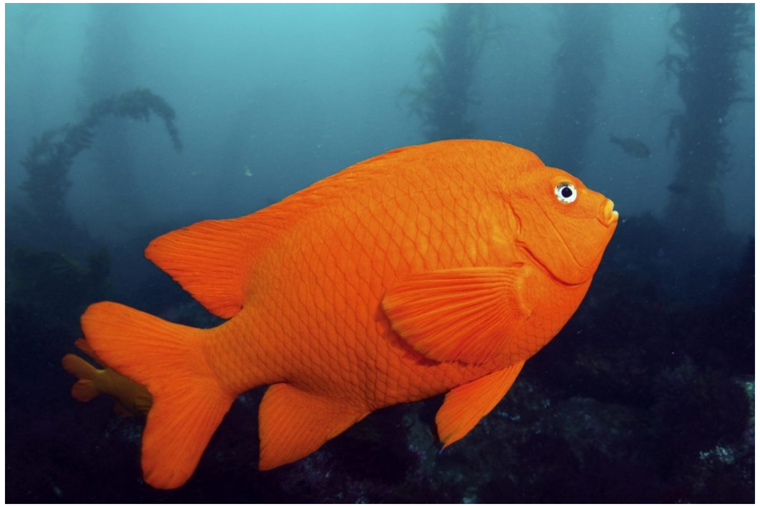
[Beverland-Campbell-Howard-Kliuchnikov 2020]: A state of T-count n can have an exponentially increasing sequence of length at most $O(n)$.

Rest of talk

<u>Lower bounds</u>

Match [Peleg, Shpilka, Volk 22] up to log factor

- $\chi\left(|H\rangle^{\otimes n}\right) \geq \Omega(n/\log n)$.
- There exists $\delta > 0$ such that $\chi_\delta\left(|H\rangle^{\otimes n}\right) \geq \Omega(\sqrt{n}/\log n)$.



<u>Upper bounds</u>

- Generic stabilizer rank

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Upper bounds: Generic stabilizer rank

- Let $\chi_n = \max\{\chi(|\psi\rangle^{\otimes n}) : |\psi\rangle \in \mathbb{C}^2\}$ be the n-th generic stabilizer rank.
- $\chi_n \geq \chi(|H\rangle^{\otimes n})$
- Fact: $\chi(|\psi\rangle^{\otimes n}) = \chi_n$ for all but finitely many $|\psi\rangle \in \mathbb{C}^2$ (up to scale).
- Proposition [Lovitz-Steffan]: $\chi_n = O(2^{n/2})$

  (Slight improvement of recent bound $O((n+1)2^{n/2})$ of [Qassim-Pashayan-Gosset 21])
- Proposition [Lovitz-Steffan]: There exists a single set of χ_n stabilizer states that can be superimposed to produce any state of the form $|\psi\rangle^{\otimes n}$.

Summary

<u>Classical simulation of Clifford+T circuits via stabilizer rank</u>

<u>Lower bounds</u>

Match [Peleg, Shpilka, Volk 22] up to log factor

- $\chi\left(|H\rangle^{\otimes n}\right) \geq \Omega(n/\log n)$.
- There exists $\delta > 0$ such that $\chi_\delta\left(|H\rangle^{\otimes n}\right) \geq \Omega(\sqrt{n}/\log n)$.

<u>Upper bounds</u>

- Generic stabilizer rank

Better bounds?

https://thumbs.dreamstime.com/b/goldfish-gold-fish-bowl-cute-cartoon-character-happy-145738808.jpg

New techniques for bounding stabilizer rank

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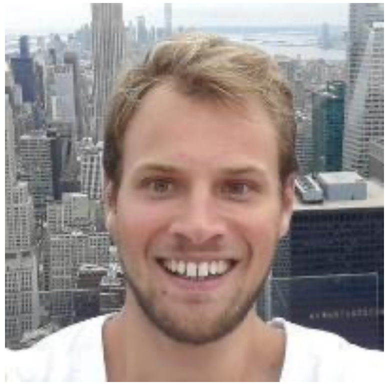

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