A splitting theorem for product tensors

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Tensor decompositions

<u>Definition:</u> Let $n \in \mathbb{N}$ and $[n] \coloneqq \{1, \dots, n\}$.

Let $\mathbb{F}$ be a field, and let X, Y, Z be $\mathbb{F}$-vector spaces of dimension at least 2.

For $T \in X \otimes Y \otimes Z$, an expression $$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z$$

is called a decomposition of T into product tensors

$$\mathrm{rank}(T) := \min\{ n : \text{there exists a decomposition of } T \text{ into } n \text{ product tensors}\}$$

Uniqueness of tensor decompositions

<u>Definition:</u> Let $n \in \mathbb{N}$ and $[n] := \{1, \ldots, n\}$.

A rank decomposition

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z$$

is called the unique (rank) decomposition of T if for any other decomposition

$$T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a \in X \otimes Y \otimes Z$$

there is a permutation $\sigma \in S_n$ such that $x_a \otimes y_a \otimes z_a = x'_{\sigma(a)} \otimes y'_{\sigma(a)} \otimes z'_{\sigma(a)}$ for all $a \in [n]$.

Application of uniqueness: Latent parameter learning

L is for *latent*

- Let A, B, C, L be finite random variables such that A, B, C are conditionally independent, i.e.

$$\Pr(a, b, c|l) = \Pr(a|l) \Pr(b|l) \Pr(c|l) \qquad \text{for all } a, b, c, l.$$

- <u>Goal:</u> Given the probability vector $\Pr(A, B, C)$, determine $\Pr(A, B, C, L)$.
- <u>Method:</u>

$$\Pr(A, B, C) = \sum_l \Pr(l) \Pr(A, B, C|l) = \sum_l \underbrace{\Pr(l) \Pr(A|l) \otimes \Pr(B|l) \otimes \Pr(C|l)}$$

… If $\Pr(A, B, C)$ has a unique decomposition, then we can recover $\Pr(A, B, C, l)$

- <u>Applications:</u> Learning mixtures of spherical gaussians, phylogenetic tree reconstruction, hidden Markov models, orbit retrieval, blind signal separation, document topic models, …

$E = \{x_a \otimes y_a \otimes z_a : a \in [n]\}.$

General flavor of most results today:

We are handed a set of product tensors E, and we want to determine properties of it.

Stronger

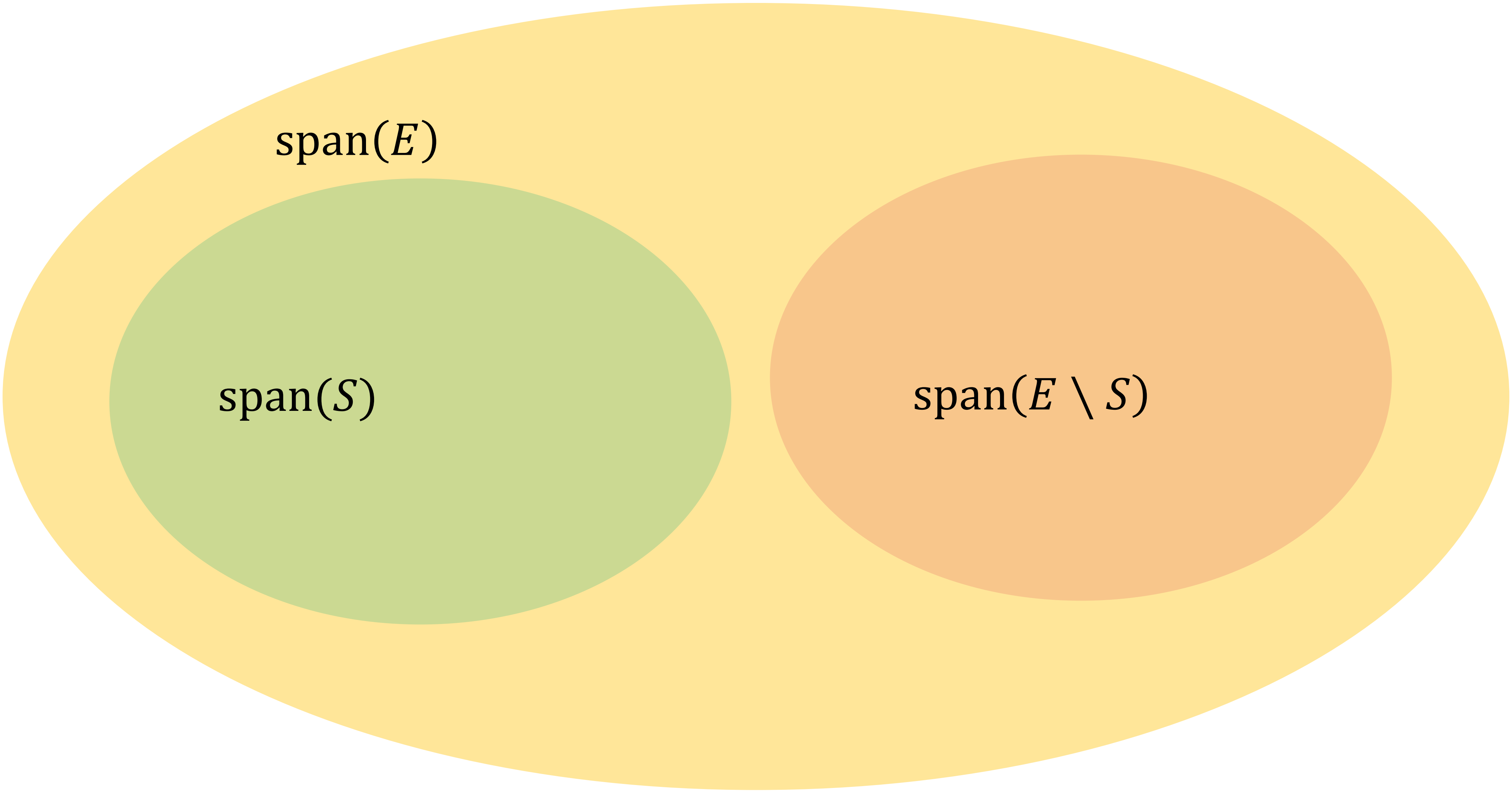
- Is $\sum(E) = \sum_{a\in[n]} x_a \otimes y_a \otimes z_a$ a unique rank decomposition?
- Is $\sum(E) = \sum_{a\in[n]} x_a \otimes y_a \otimes z_a$ a rank decomposition?
- Is every linear combination of E that produces a product tensor trivial (of the form $\alpha(x_a \otimes y_a \otimes z_a)$)?
- Is E linearly independent?
- Does E split?

Splitting

Definition: A set of vectors $E = \{v_1, \ldots, v_n\}$ splits if there exists a non-trivial subset $S \subseteq E$ such that

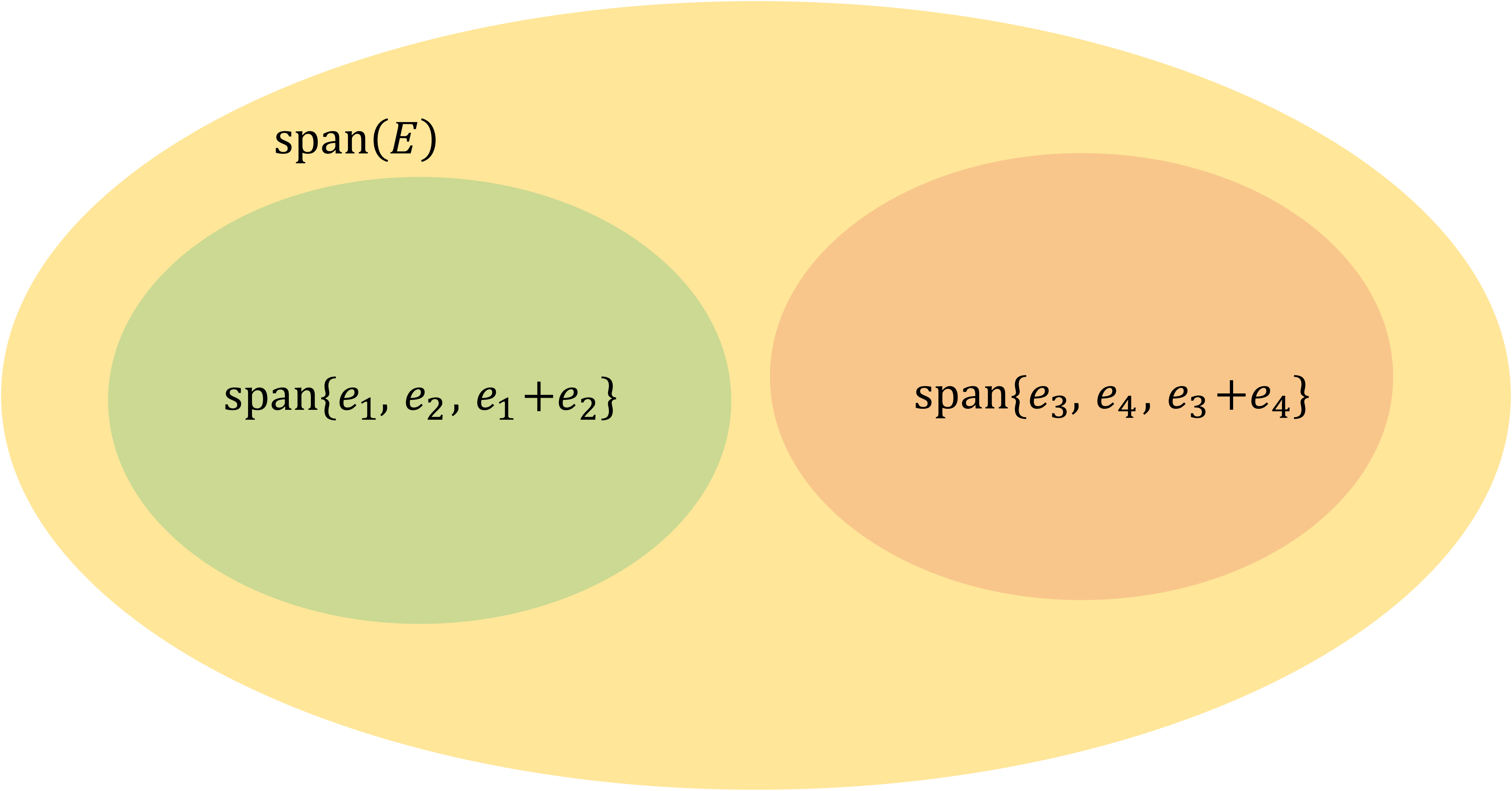
$$\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\} \tag{2}$$

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$



E splits if there exists $S \subseteq E$ such that $\mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$

$E = \{e_1, e_2, e_1 + e_2, e_3, e_4, e_3 + e_4\}$



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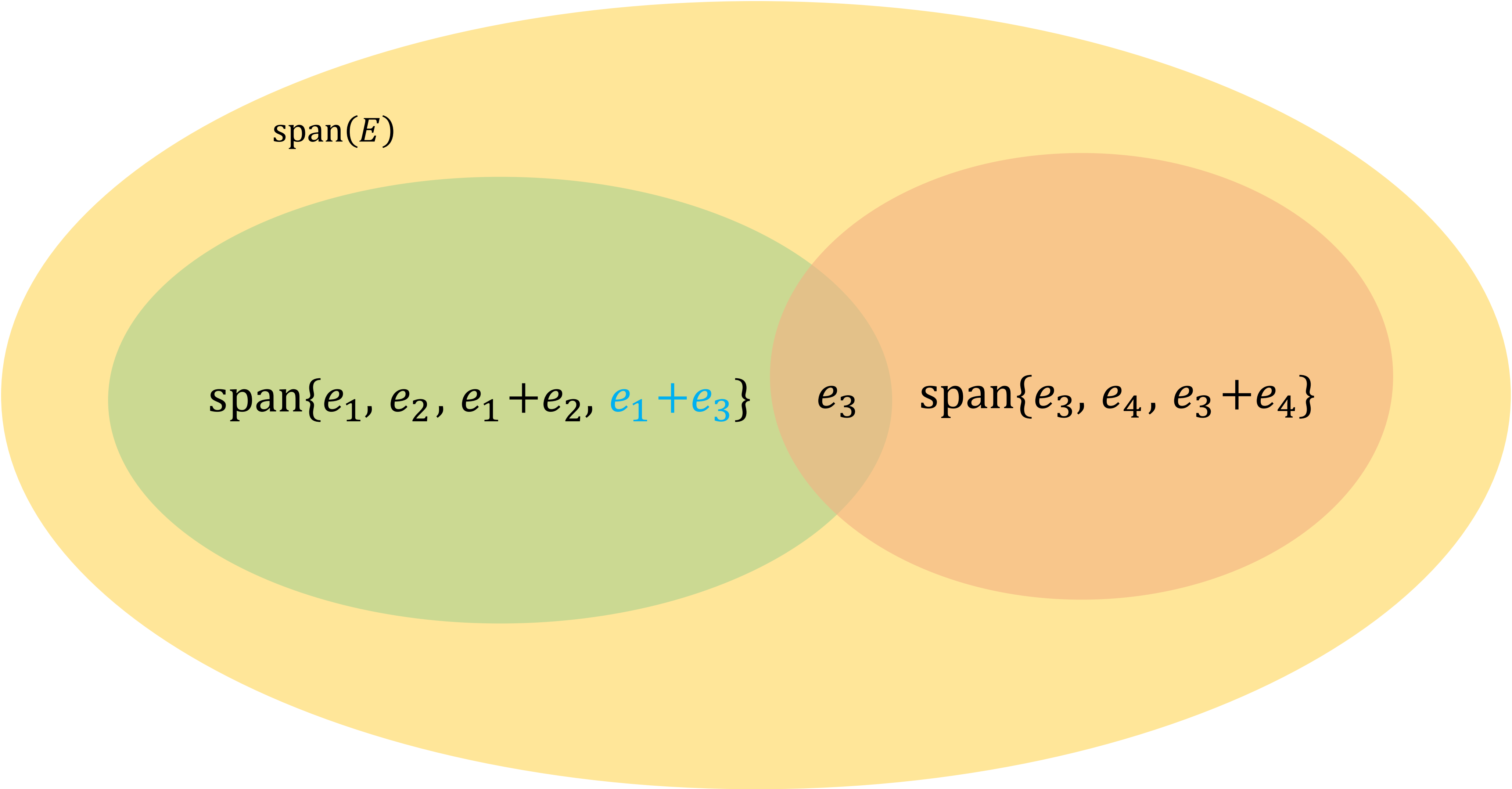

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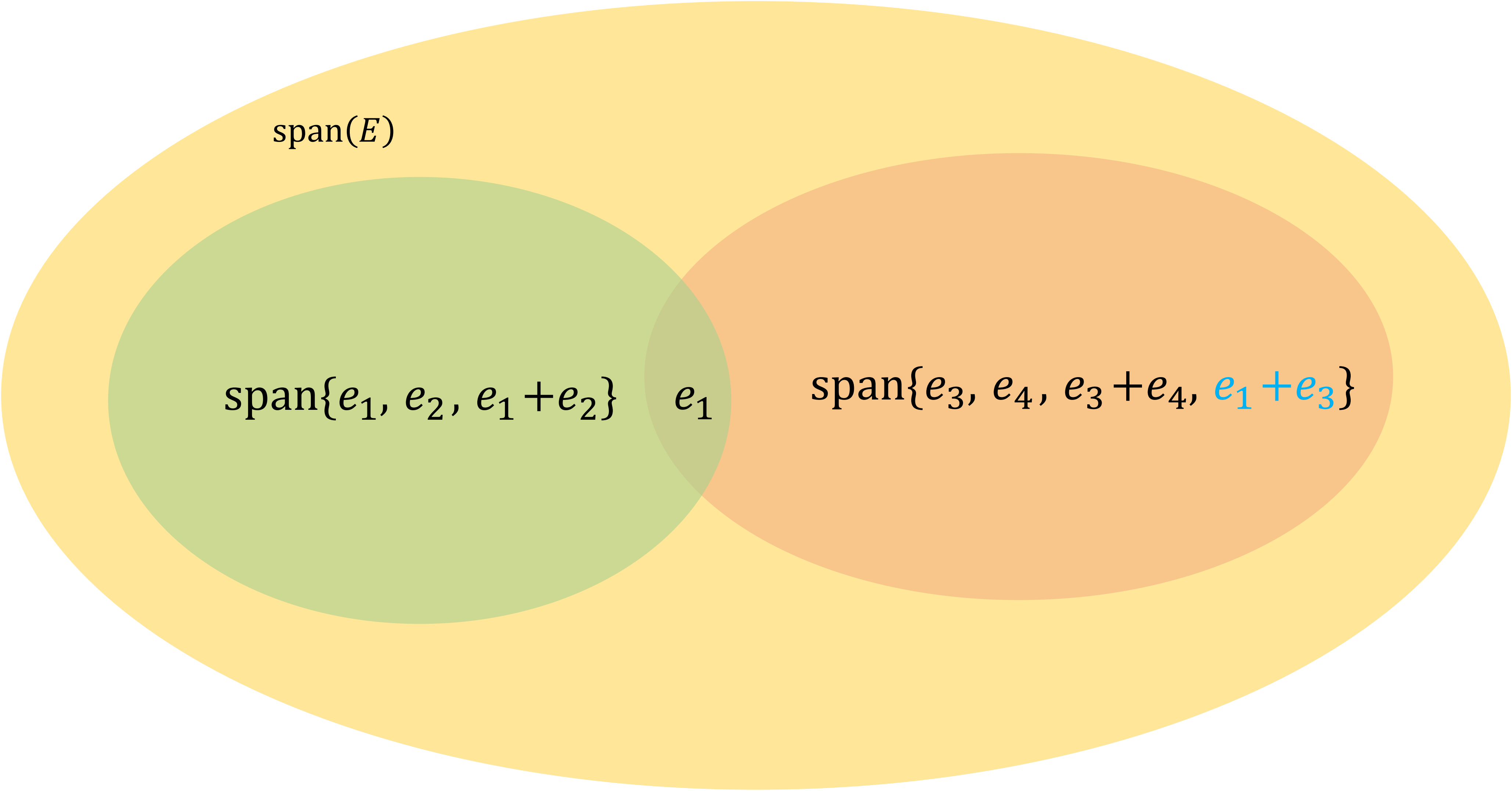

Splitting theorem

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Splitting theorem [L-Petrov]: Let $E = \{x_a \otimes y_a : a \in [n]\}$.

If

$$\mathrm{dimspan}(E) \leq d_x^{[n]} + d_y^{[n]} - 2$$

$$d_x^{[n]} = \mathrm{dimspan}\{x_1, \ldots, x_n\}$$

then E splits.

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If

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$$d_x^{[n]} = \mathrm{dimspan}\{x_1, \ldots, x_n\}$$

Corollary 1: If

$$n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then E splits.

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Linear independence

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Corollary 2: If $|S| \leq d_x^{[S]} + d_y^{[S]} + d_z^{[S]} - 2$ for all $S \subseteq [n]$ of size $|S| \geq 2$, then E is linearly independent.

$d_x^{[S]} = \mathrm{dimspan}\{x_a | a \in S\}$

Example: $E = \{|100\rangle, |010\rangle, |001\rangle\}$

- $\sum(E) = W = |100\rangle + |010\rangle + |001\rangle$
- $n = 3$
- $S = [n] \Rightarrow |S| = 3 \leq 2 + 2 + 2 - 2 = 4$
- $S = \{1,2\} \Rightarrow |S| = 2 \leq 2 + 2 + 1 - 2 = 3$ $S = \{2,3\}, S = \{1,3\}$ similar
- $\Rightarrow E$ is linearly independent

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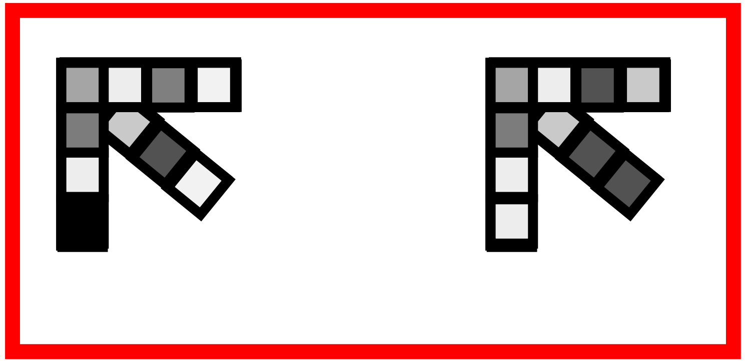
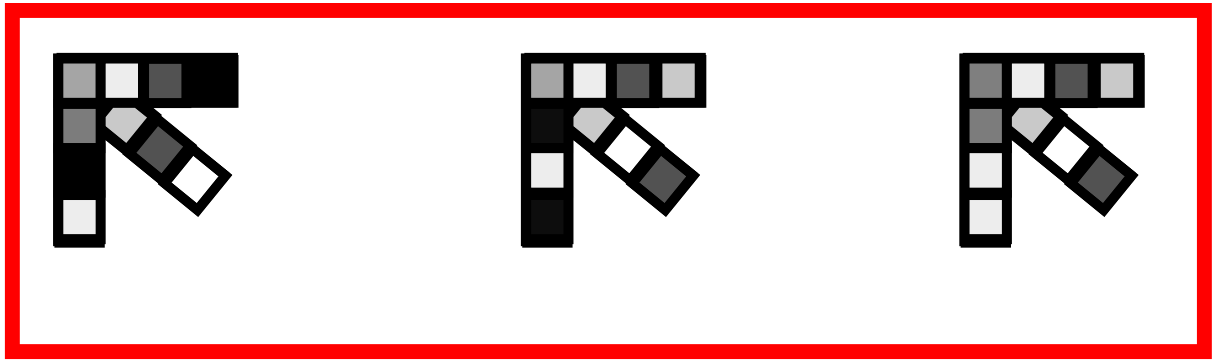
$E =$ 


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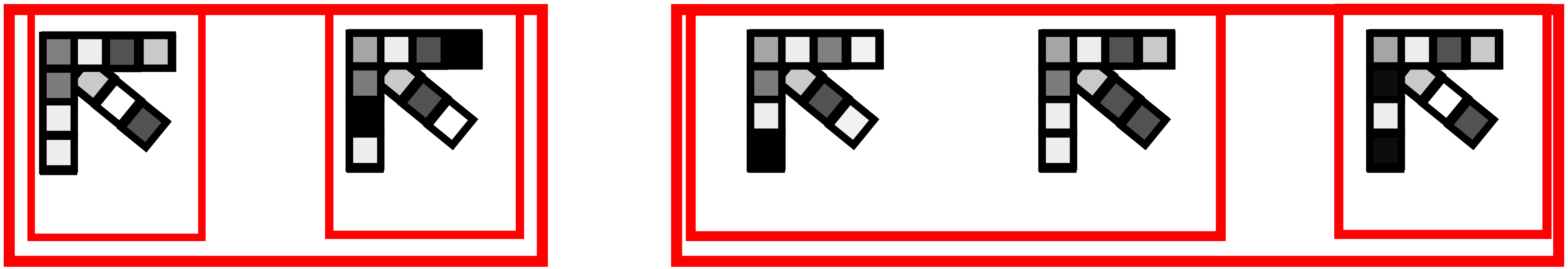
$E =$ 

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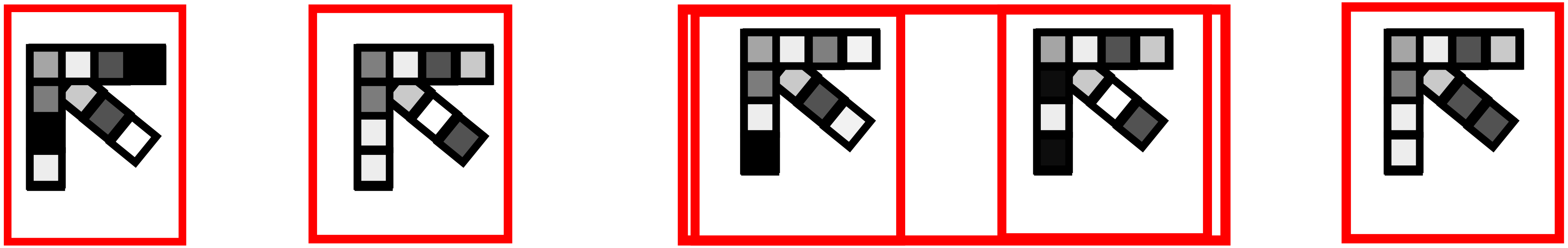
$E =$ 

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Uniqueness

$E = \{x_a \otimes y_a \otimes z_a : a \in [n]\}.$

Theorem [L-Petrov]: If for every subset $S \subseteq [n]$ of size $|S| \geq 2$, it holds that

$$2|S| \leq d_x^S + d_y^S + d_z^S - 2,$$

$d_x^S = \mathrm{dimspan}\{x_a : a \in S\}$

then $\sum(E)$ is a unique rank decomposition.

(Strengthens a uniqueness theorem of Joseph Kruskal)

Example: W state

Theorem: If for all $S \subseteq [n]$ of size $|S| \geq 2$, it holds that $2|S| \leq d_x^S + d_y^S + d_z^S - 2$, then $\sum(E)$ is a unique rank decomposition.

- $E = \{|100\rangle, |010\rangle, |001\rangle\}$, so $\sum(E) = W$.
- $n = 3$
- $S = [n] \Rightarrow 2|S| = 6 \nleq 2 + 2 + 2 - 2 = 4$

…In fact, the W state has other rank decompositions besides

$$\sum(E) = |100\rangle + |010\rangle + |001\rangle$$

Example: GHZ state

Theorem: If for all $S \subseteq [n]$ of size $|S| \geq 2$, it holds that $2|S| \leq d_x^S + d_y^S + d_z^S - 2$, then $\sum(E)$ is a unique rank decomposition.

- $E = \{|111\rangle, \dots, |nnn\rangle\}$
- $\sum(E) = GHZ_n = \sum_{a \in [n]} |aaa\rangle$
- For all $S \subseteq [n]$ of size $|S| \geq 2$,

$$2|S| \leq d_x^S + d_y^S + d_z^S - 2 = 3|S| - 2$$

…so $\sum(E) = \sum_{a \in [n]} |aaa\rangle$ is the unique rank decomposition of the GHZ state.

$$E = \{x_a \otimes y_a \otimes z_a : a \in [n]\}.$$

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Interpolating statement

$$E = \{x_a \otimes y_a \otimes z_a : a \in [n]\}.$$

Interpolating statement

<u>Theorem:</u> Let $r \in \{0,1,\ldots,n\}$. If for all $S \subseteq [n]$ of size $|S| \geq 2$, it holds that

$$|S| + \min\{|S|, r\} \leq d_x^S + d_y^S + d_z^S - 2,$$

then the only linear combinations

$$T_\alpha = \sum_{a \in [n]} \alpha_a (x_a \otimes y_a \otimes z_a)$$

for which $\text{rank}(T_\alpha) = r$ are those for which $\#\{a : \alpha_a \neq 0\} = r$.

… Furthermore, this is the unique rank decomposition of T_α.

$E = \{x_a \otimes y_a \otimes z_a : a \in [n]\}.$

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Stronger than unique?!?!?!

Stronger

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Example: GHZ state (Stronger than unique!)

- $E = \{|1\rangle^{\otimes m}, \ldots, |n\rangle^{\otimes m}\}$

- <u>Corollary:</u> There are no other decompositions of

$$\Sigma(E) = \sum_{a \in [n]} |a\rangle^{\otimes m}$$

into at most $n + m - 3$ *symmetric* product tensors.

Greater than n when $m > 3$!

Quantum application: Absolutely entangled sets

Question: Given unit vectors $v_1, \dots, v_n \in \mathbb{C}^d$, does there exist a positive integer m and an isometry $U: \mathbb{C}^d \to (\mathbb{C}^2)^{\otimes m}$ for which $Uv_a \in (\mathbb{C}^2)^{\otimes m}$ is a product tensor for all $a \in [m]$?

If "yes": Protocol using $v_1, \dots, v_n$ could be converted into a protocol that uses tensor products of qubits.

If "no": The set $\{v_1, \dots, v_n\}$ is "absolutely entangled."

Corollary to splitting theorem: If $\{v_1, \dots, v_n\}$ does not split, then we can assume $m \leq n - 2$

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Theorem [Lovitz, 2019]:

n	1	2	3	4	≥ 5
Does such a U always exist?	Yes	Yes	Yes	?	No