New techniques for bounding stabilizer rank

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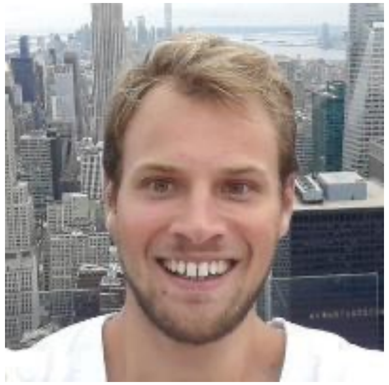

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Computational basis

- Let $\{|0\rangle, |1\rangle\} \subseteq \mathbb{C}^2$ be the computational basis for $\mathbb{C}^2$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Let $\{|x\rangle := |x_1\rangle \otimes \cdots \otimes |x_n\rangle : x \in \{0,1\}^n\} \subseteq (\mathbb{C}^2)^{\otimes n}$ be the computational basis for $(\mathbb{C}^2)^{\otimes n}$

$$|00\rangle := |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad |01\rangle := |0\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

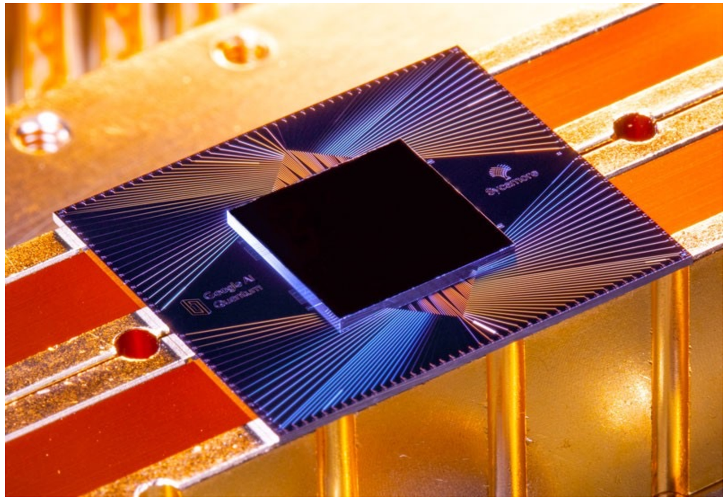
$$|10\rangle := |1\rangle \otimes |0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad |11\rangle := |1\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- A state is a unit vector in $(\mathbb{C}^2)^{\otimes n}$ (mod phase, i.e. an element of $\mathbb{P}^{2^n-1}$).
- We often omit normalization.
- States in $\mathbb{C}^2$ are called qubits.
- States in $(\mathbb{C}^2)^{\otimes n}$ are called n-qubit states.
- States are denoted $|\psi\rangle, |\phi\rangle$, etc.
- $\langle\psi|$ denotes conjugate-transpose of $|\psi\rangle$

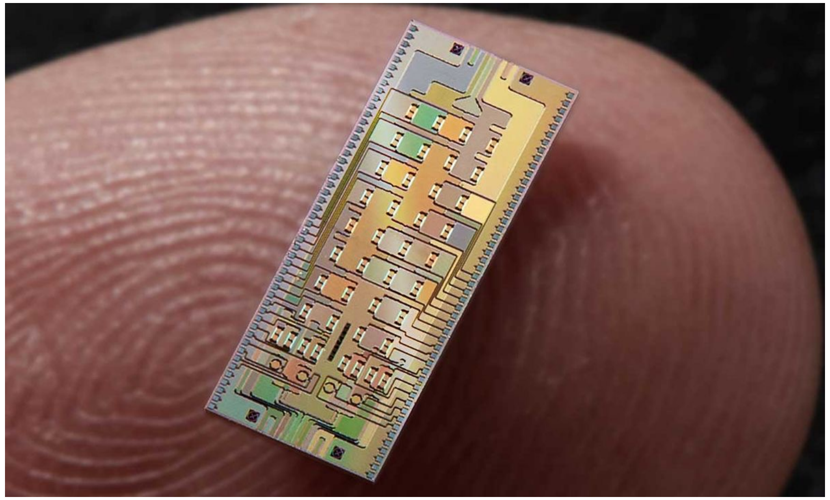
Quantum circuits

<u>General framework:</u>

1. Prepare a computational basis state $|0\cdots 0\rangle \in (\mathbb{C}^2)^{\otimes n}$.
2. Apply a unitary matrix to obtain $|\psi\rangle \coloneqq U|0\cdots 0\rangle$
3. Measure in the computational basis. For $x \in \{0,1\}^n$, $p(x) = |\langle x|\psi\rangle|^2$



Google Sycamore superconducting qubit chip.
n=53 qubits (with errors!!)



Xanadu X8 photonic chip
n=8 qubits (with errors!!)

Quantum circuits

General framework:

1. Prepare a computational basis state $|0\cdots0\rangle \in (\mathbb{C}^2)^{\otimes n}$.
2. Apply a unitary matrix to obtain $|\psi\rangle := U|0\cdots0\rangle$
3. Measure in the computational basis. For $x \in \{0,1\}^n$, $p(x) = |\langle x|\psi\rangle|^2$.

The measurement destroys the state!

Need $\Omega(2^n)$ repetitions to approximate p.

Subtle power of quantum computer: You can sample from $p \in \mathbb{R}_+^{2^n}$

Quantum circuits

<u>General framework:</u>

1. Prepare a computational basis state $|0 \cdots 0\rangle \quad \in (\mathbb{C}^2)^{\otimes n}$.
2. Apply a unitary matrix to obtain $|\psi\rangle := U|0 \cdots 0\rangle$
3. Partially measure. For $x \in \{0,1\}^k$, $p(x) = ||(\langle x| \otimes \mathbb{I})|\psi\rangle\,||^2$

$k \leq n$

<u>Contraction:</u>

Defined as $(\langle x_1 \cdots x_k| \otimes \mathbb{I})|y_1 \cdots y_n\rangle := \delta_{x_1,y_1} \cdots \delta_{x_k,y_k}|y_{k+1} \cdots y_n\rangle \in (\mathbb{C}^2)^{\otimes n-k}$

on computational basis vectors $|y_1 \cdots y_n\rangle$, and extended linearly.

Quantum circuits

<u>General framework:</u>

1. Prepare a computational basis state $|0\cdots 0\rangle \quad \in (\mathbb{C}^2)^{\otimes n}$.
2. Apply a unitary matrix to obtain $|\psi\rangle := \ U|0\cdots 0\rangle$
3. Partially measure. For $x \in \{0,1\}^k$, $p(x) = ||(\langle x| \otimes \mathbb{I})|\psi\rangle\,||^2$

$k \leq n$

Partial measurement only partially destroys the state.

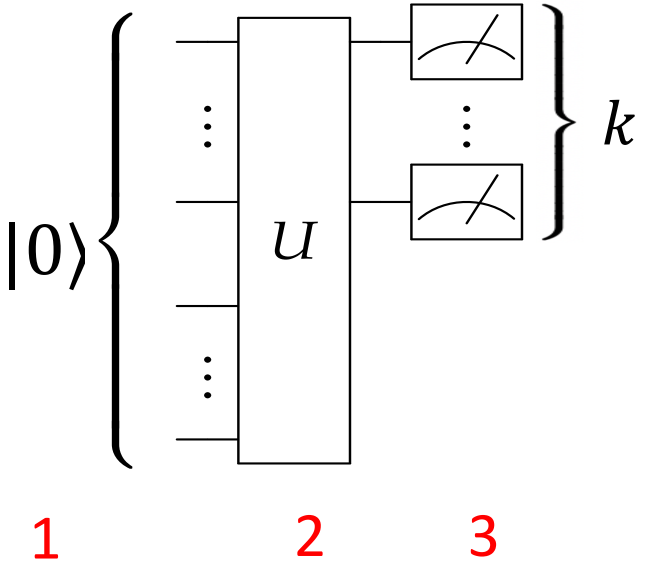
Leftover state is $\dfrac{1}{\sqrt{p(x)}}(\langle x| \otimes \mathbb{I})U|0\cdots 0\rangle \in (\mathbb{C}^2)^{\otimes n-k}$.

Normalization

Quantum circuits

<u>General framework:</u>

1. Prepare a computational basis state $|0\rangle \in (\mathbb{C}^2)^{\otimes n}$.
2. Apply a unitary matrix $U|0\rangle$
3. Measure in the computational basis. For $x \in \{0,1\}^k$, $p(x) = ||(\langle x| \otimes \mathbb{I})U|0\rangle||^2$



This talk: Classical simulation of Clifford+T circuits via stabilizer rank

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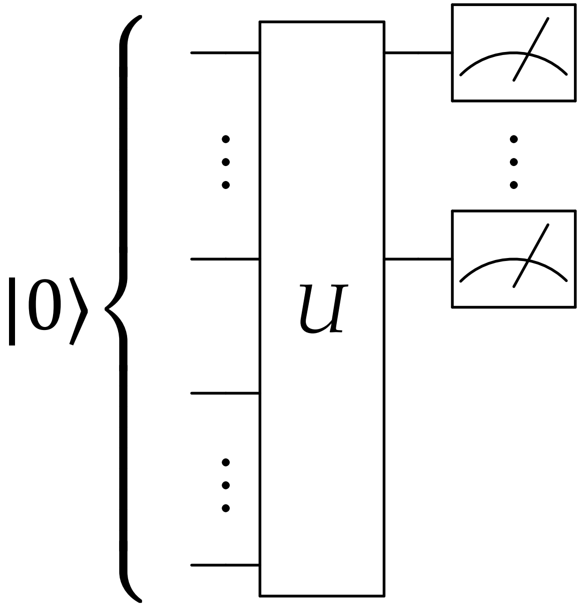
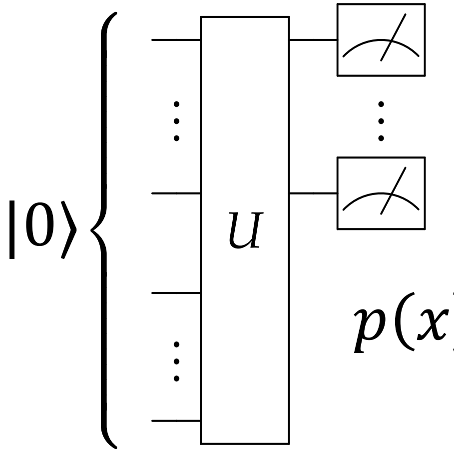
Classical simulation of quantum circuits

Question: Given a classical description of a quantum circuit



Can it be simulated efficiently on a classical computer?

Types of simulation



$x \in \{0,1\}^k$

$p(x) = ||(\langle x| \otimes \mathbb{I})U|0\rangle^{\otimes n}||^2$

- Strong simulation:

  Compute $p(x)$ for all $x \in \{0,1\}^k$.

- ϵ-Strong simulation:

  Find a probability vector $\tilde{p}$ such that

  $$(1-\epsilon)p(x) \le \tilde{p}(x) \le (1+\epsilon)p(x) \quad \text{for all } x \in \{0,1\}^k.$$

- Weak simulation:

  Sample elements of $x \in \{0,1\}^k$ from a probability distribution $\tilde{p}$ such that

  $$\|\tilde{p} - p\|_1 \le \epsilon$$

This talk: Classical simulation of Clifford+T circuits via stabilizer rank

Clifford circuits

The Clifford group is the group of unitaries $U: (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$ generated by the Clifford gates

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

… and global phases $U(1) = \{e^{i\theta} : \theta \in [0,2\pi)\}$

The Pauli group on $\mathbb{C}^2$ is the group of unitaries $U: \mathbb{C}^2 \to \mathbb{C}^2$ generated by the Pauli gates

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad iI_2 = i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

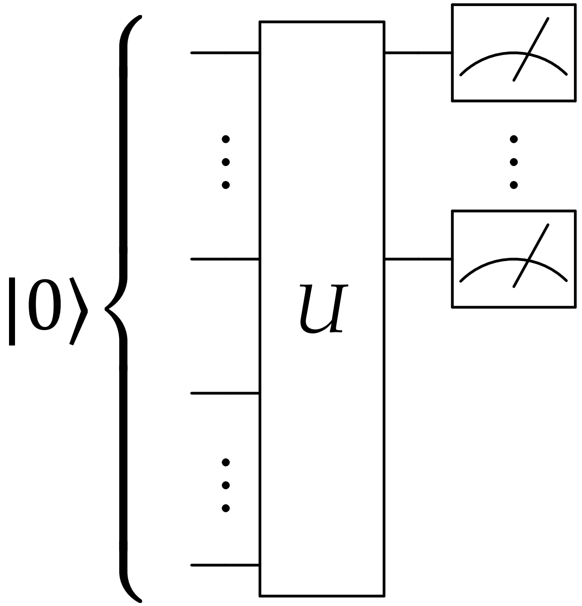
- As an abstract group, the Pauli group on $\mathbb{C}^2$ is the central product $C_4 \circ D_4$.
- The Clifford group is the normalizer of the n-fold tensor product of the unitary Pauli group.
- Open problem: Character table for Clifford group?

Why Clifford circuits?

1. Implementation
2. Error correction
3. Clifford + any other gate is dense in the unitary group
4. Standard set of gates

Classical simulation of Clifford circuits

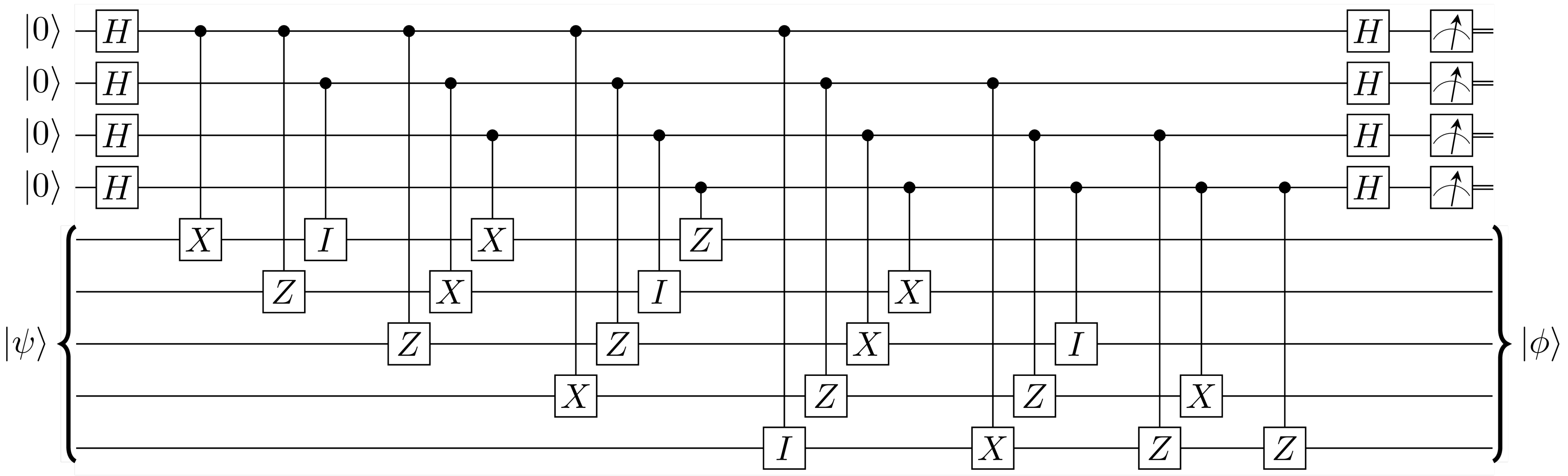
Question: Given a classical description of a Clifford circuit



Can it be simulated efficiently on a classical computer?

Classical simulation of Clifford circuits

Question: Given a classical description of a Clifford circuit



Can it be simulated efficiently on a classical computer?

[Gottesman-Knill 98]: Yes. Clifford circuits can be efficiently simulated.

... Strongly, weakly, and ϵ-strongly

... Even the leftover state $\frac{1}{\sqrt{p(x)}}(\langle x| \otimes \mathbb{I})U|0 \cdots 0\rangle$

can be computed (and represented) efficiently!

Clifford+T circuits

The Clifford+T group is the unitary group $U: (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$ generated by the Clifford gates

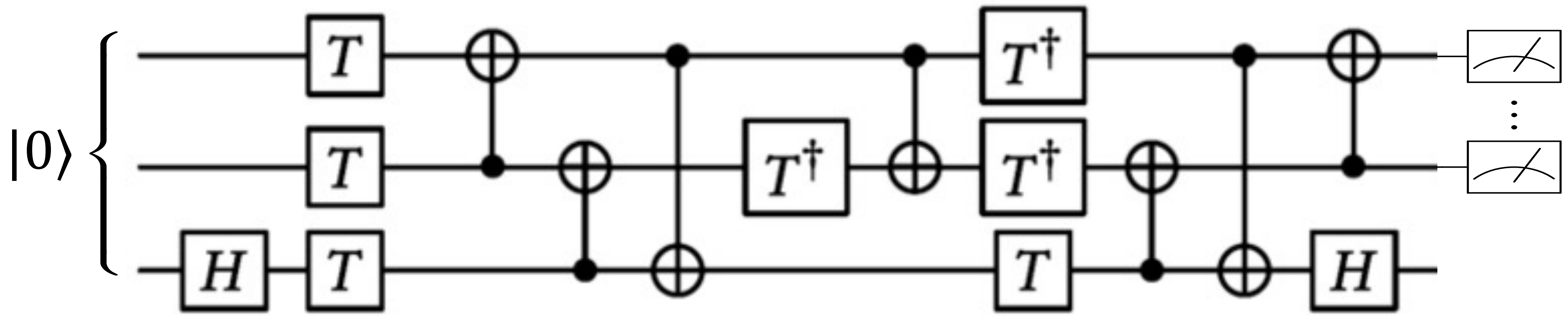
$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

and T-gates

$$\mathrm{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}.$$

Classical simulation of Clifford+T circuits

Question: Given a classical description of a Clifford+T circuit



Can it be simulated efficiently on a classical computer?

Quantum circuits

General framework:

1. Prepare a computational basis state $|0 \cdots 0\rangle \in (\mathbb{C}^2)^{\otimes n}$.
2. Apply a small number of Clifford+T gates $U_1 U_2 \ldots U_s |0 \cdots 0\rangle$
3. Measure in the computational basis. For $x \in \{0,1\}^n$, $p(x) = |\langle x|U_1 U_2 \ldots U_s|0 \cdots 0\rangle|^2$.

Subtle power of quantum computer: Can apply $U_1 U_2 \ldots U_s$ efficiently, and sample from $p \in \mathbb{R}_+^{2^n}$.

This talk: Classical simulation of Clifford+T circuits via stabilizer rank

Stabilizer rank

- A state $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ is a stabilizer state if $|\phi\rangle = U|0\rangle^{\otimes n}$ for some Clifford circuit U.

- The stabilizer rank of a state $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$, denoted $\chi(|\psi\rangle)$, is the minimum number r for which

$$|\psi\rangle = \sum_{i=1}^{r} c_i|\phi_i\rangle$$

for some $c_i \in \mathbb{C}$ and $|\phi_i\rangle$ stabilizer states.

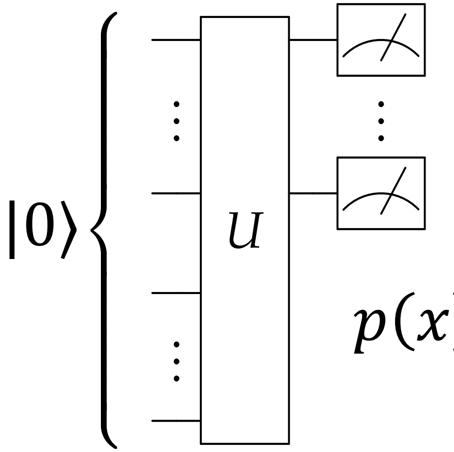
- The δ-approximate stabilizer rank of $|\psi\rangle$ is

$$\chi_\delta(|\psi\rangle) = \min\{\chi(|\mu\rangle): \||\psi\rangle - |\mu\rangle\| \leq \delta\}.$$

Stabilizer rank

$|H\rangle = |0\rangle + (1+\sqrt{2})|1\rangle \approx |0\rangle + 2.41|1\rangle$

[Bravyi-Smith-Smolin 16, Bravyi-Gosset 16]: A Clifford+T circuit with n T-gates can be simulated…



$x \in \{0,1\}^k$

$p(x) = ||(\langle x| \otimes \mathbb{I})U|0\rangle^{\otimes n}||^2$

- Strongly with cost quadratic in $\chi(|H\rangle^{\otimes n})$.

  Compute $p(x)$ for all $x \in \{0,1\}^k$

- ϵ-Strongly with cost linear in $\chi(|H\rangle^{\otimes n})$.

  Find a probability vector $\tilde{p}$ such that $(1-\epsilon)p(x) \le \tilde{p}(x) \le (1+\epsilon)p(x)$

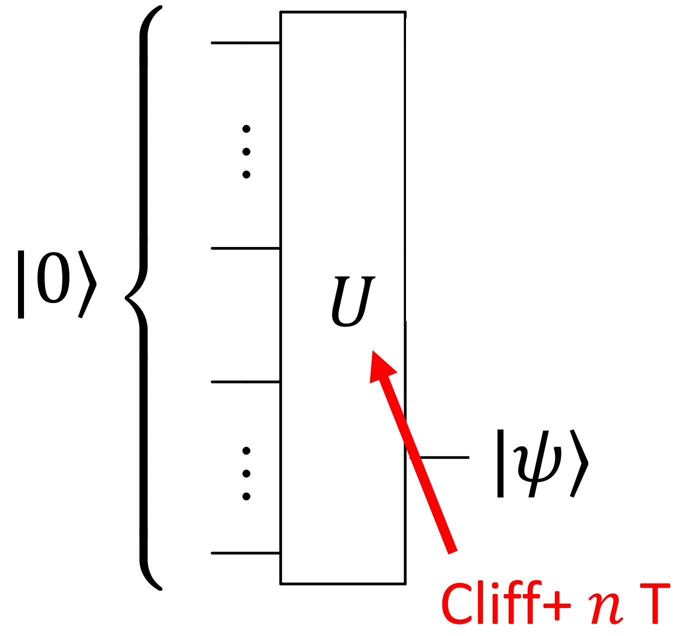
- Weakly with cost linear in $\chi_\delta(|H\rangle^{\otimes n})$.

  Sample elements of $x \in \{0,1\}^k$ from a probability distribution $\tilde{p}$ such that

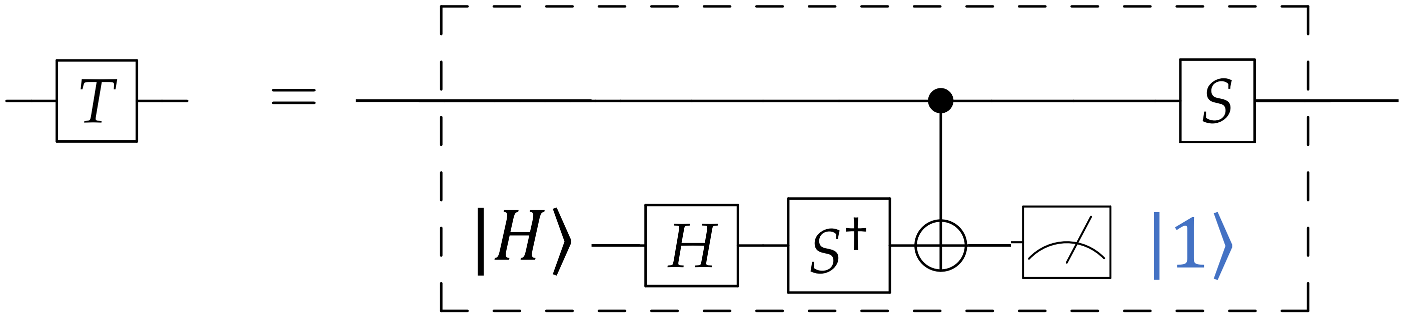
  $$\|\tilde{p} - p\|_1 \le \epsilon$$

Proof idea

[GK98]: Clifford circuits can be efficiently simulated.



$$-\boxed{T}- = (\mathbb{I} \otimes \langle 1|) U_{\text{Cliff}} (\mathbb{I} \otimes |H\rangle)$$



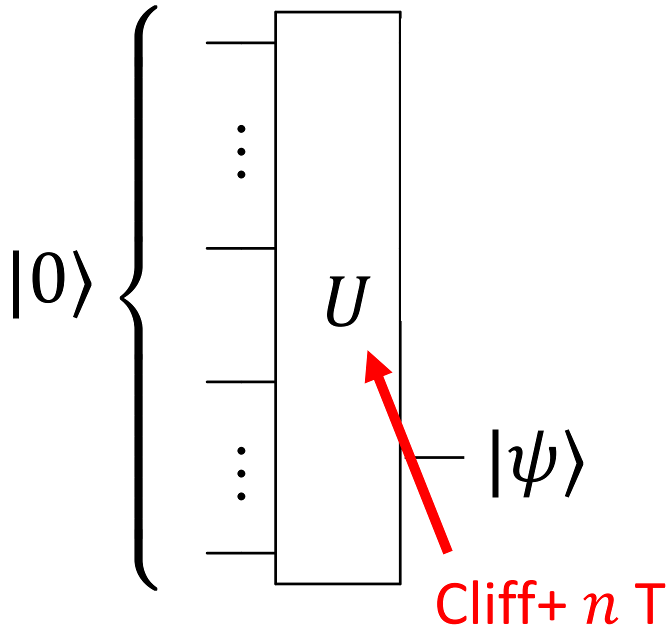
$$\text{So } T|0\rangle = (\mathbb{I} \otimes \langle 1|) U_{\text{Cliff}} (|0\rangle \otimes |H\rangle)$$

$$= (\mathbb{I} \otimes \langle 1|) U_{\text{Cliff}} (|0\rangle \otimes |0\rangle)$$

$$+ (1+\sqrt{2}) (\mathbb{I} \otimes \langle 1|) U_{\text{Cliff}} (|0\rangle \otimes |1\rangle)$$

Proof idea

[GK98]: Clifford circuits can be efficiently simulated.



$$-\boxed{T}- \;=\; (\mathbb{I} \otimes \langle 1|) U_{\text{Cliff}} (\mathbb{I} \otimes |H\rangle)$$

Let $r = \chi\left(|H\rangle^{\otimes n}\right)$ and $|H\rangle^{\otimes n} = \sum_{i=1}^{r} c_i |\phi_i\rangle$.

$$|\psi\rangle = (\mathbb{I} \otimes \langle 1 \ldots 1|) U_{\text{Cliff}} (|0\rangle \otimes |H\rangle^{\otimes n})$$
$$= \sum_{i=1}^{r} c_i (\mathbb{I} \otimes \langle 1 \ldots 1|) \underbrace{U_{\text{Cliff}} (|0\rangle \otimes |\phi_i\rangle)}_{= U_{\text{Cliff}}^{(i)} |0\rangle}$$

By [GK98], can simulate each efficiently.

Known bounds on stabilizer rank

$\chi(|H\rangle^{\otimes n})$

- [Huang-Newman-Szegedy 20]: $\chi(|H\rangle^{\otimes n})$ super-polynomial unless P=NP.
- [Bravyi-Smith-Smolin 16]: $\chi(|H\rangle^{\otimes n}) \geq \Omega(\sqrt{n})$.
- [Peleg-Shpilka-Volk 21]: $\chi(|H\rangle^{\otimes n}) \geq \Omega(n)$.
- [Qassim-Pashayan-Gosset 21]: $\chi(|H\rangle^{\otimes n}) \leq 2^{\alpha n}$, where $\alpha = \frac{1}{4}\log_2(3)$.

$\chi_\delta(|H\rangle^{\otimes n})$

This talk: Alternate proofs up to log factor

- [Peleg-Shpilka-Volk 21]:

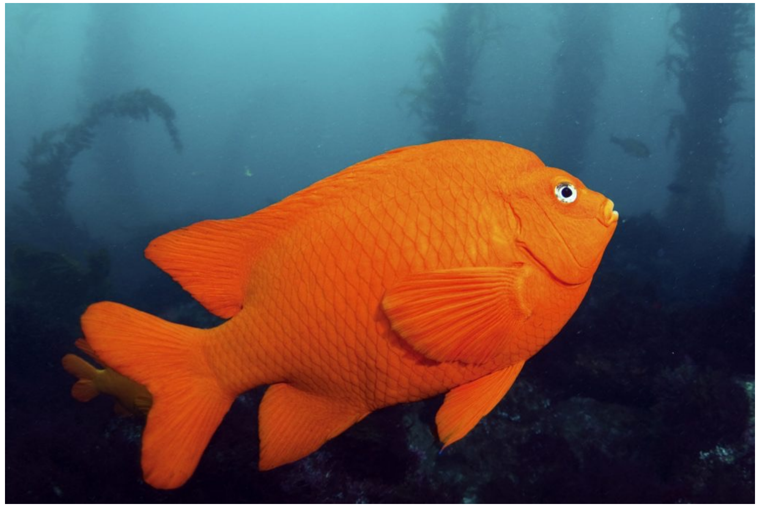
  There exists $\delta > 0$ such that $\chi_\delta(|H\rangle^{\otimes n}) \geq \Omega(\sqrt{n}/\log n)$
- [Bravyi-Gosset 16]: $\chi_\delta\left(|H\rangle^{\otimes n}\right) \leq O\left(\frac{1}{\delta^2} 2^{\alpha m}\right)$, where $\alpha \approx 0.228$.

Rest of talk

<u>Lower bounds</u>

Match [Peleg, Shpilka, Volk 22] up to log factor

- $\chi\left(|H\rangle^{\otimes n}\right) \geq \Omega(n/\log n)$.
- There exists $\delta > 0$ such that $\chi_\delta\left(|H\rangle^{\otimes n}\right) \geq \Omega(\sqrt{n}/\log n)$.



<u>Upper bounds</u>

- Generic stabilizer rank

Image: https://fishingbooker.com/blog/california-state-fish-golden-trout-garibaldi/

Fact: If $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ is a stabilizer state, then the coordinates of $|\phi\rangle$ are $\{0, \pm 1, \pm i\}$ (up to normalization).

Theorem [Dehaene, De Moor 03]:

$|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ is a stabilizer state $\Leftrightarrow$ $|\phi\rangle = \sum_{x\in A} i^{l(x)}(-1)^{q(x)}|x\rangle$,

Identify $\{0,1\}^n$ with $\mathbb{F}_2^n$

where $A \subseteq \mathbb{F}_2^n$ is an affine linear subspace

$l: \mathbb{F}_2^n \to \mathbb{F}_2$ is a linear function

$q: \mathbb{F}_2^n \to \mathbb{F}_2$ is a quadratic polynomial

Most results on stabilizer rank, and modern proofs of [GK98], use this characterization!

Subset-sum representations

- Let $\alpha \in \mathbb{C}^k, \beta \in \mathbb{C}^r$. We say β is a subset-sum representation of α if each α_j is equal to the sum of some subset of $\{\beta_1, \dots, \beta_r\}$.

- <u>Example:</u> $\beta = (1,2)$ is a subset-sum representation of $\alpha = (1,2,3)$.

- <u>Example:</u> If $|\psi\rangle = \sum_{i=1}^{r} c_i |\phi_i\rangle$ is a stabilizer decomposition, then

$$\beta = (c_1, \dots, c_r, -c_1, \dots, -c_r, ic_1, \dots, ic_r, -ic_1, \dots, -ic_r) \in \mathbb{C}^{4r}$$

is a subset-sum representation of $|\psi\rangle$.

$|\phi_i\rangle$ stabilizer $\Rightarrow$coordinates are in $\{0, \pm 1, \pm i\}$

$\Rightarrow \chi(|\psi\rangle) \geq \frac{1}{4} \cdot$ (the size of the smallest subset-sum rep of $|\psi\rangle$)

Lower bounds on the size of a subset-sum rep

- Let $\alpha \in \mathbb{C}^k, \beta \in \mathbb{C}^r$. We say β is a subset-sum representation of α if each α_j is equal to the sum of some subset of $\{\beta_1, \ldots, \beta_r\}$.

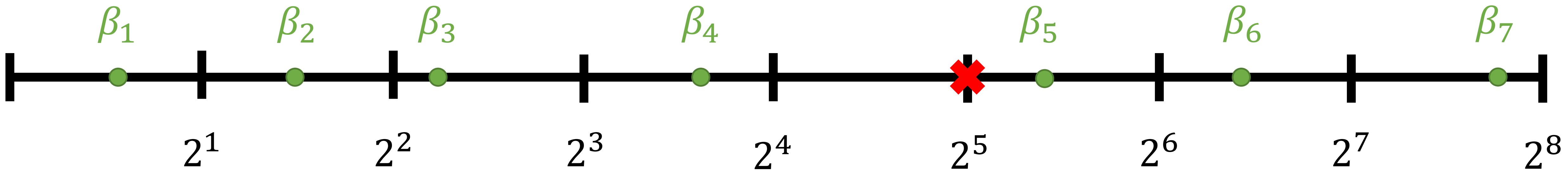
- Trivially, $\mathrm{r} \geq \log_2 k$, since $\{\beta_1, \ldots, \beta_r\}$ has just 2^r different subsets.

α exponentially increasing

- Theorem [Moulton 01]: If $2|\alpha_j| \leq |\alpha_{j+1}|$ for all $j \in \{1, \ldots, k-1\}$, then $r \geq k/\log_2 k$.

Linear in k, instead of logarithmic!

- Example: If $\alpha = (2^1, 2^2, \ldots, 2^k)$, then $r \geq k/\log_2 k$



Lower bounds on the size of a subset-sum rep

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Linear in k, instead of logarithmic!

- Example: If $\alpha = (2^1, 2^2, \ldots, 2^k)$, then $r \geq k/\log_2 k$

- Theorem [Lovitz-Steffan]: If the coordinates of $|\psi\rangle$ contain an exponentially increasing sequence of length k, then $\chi(|\psi\rangle) \geq k/(4\log_2 k)$.

Lower bound on stabilizer rank

- Theorem [Lovitz-Steffan]: If the coordinates of $|\psi\rangle$ contain an exponentially increasing sequence of length k, then $\chi(|\psi\rangle) \geq k/(4\log_2 k)$.

Corollary [Lovitz-Steffan]: $\chi(|H\rangle^{\otimes n}) \geq n/(4\log_2 n)$.

Proof: Since $|H\rangle \approx |0\rangle + 2.41|1\rangle$,

$$|H\rangle^{\otimes n} \approx |0\cdots 0\rangle + (2.41)(|0\cdots 01\rangle + \cdots + |10\cdots 0\rangle) + \cdots + (2.41)^n|1\cdots 1\rangle.$$

$\Rightarrow |H\rangle^{\otimes n}$ contains the exponentially increasing sequence $(2.41, 2.41^2, \ldots, 2.41^n)$

$\Rightarrow \chi(|H\rangle^{\otimes n}) \geq n/(4\log_2 n)$ by boxed theorem.

$\blacksquare$

Lower bound on approximate stabilizer rank

- The δ-approximate stabilizer rank of a normalized state $|\psi\rangle$ is
$$\chi_\delta(|\psi\rangle) = \min\{\chi(|\mu\rangle) : \||\psi\rangle - |\mu\rangle\| \leq \delta\}.$$

- Theorem [Lovitz-Steffan]: There exists $\delta > 0$ for which $\chi_\delta(|H\rangle^{\otimes n}) \geq \sqrt{n}/(4\log_2 \sqrt{n})$.

Proof sketch: Show that for δ small enough, any state that is δ-close to $|H\rangle^{\otimes n}$ must contain an exponentially increasing sequence of length $\sqrt{n}$ (Use De Moivre-Laplace).

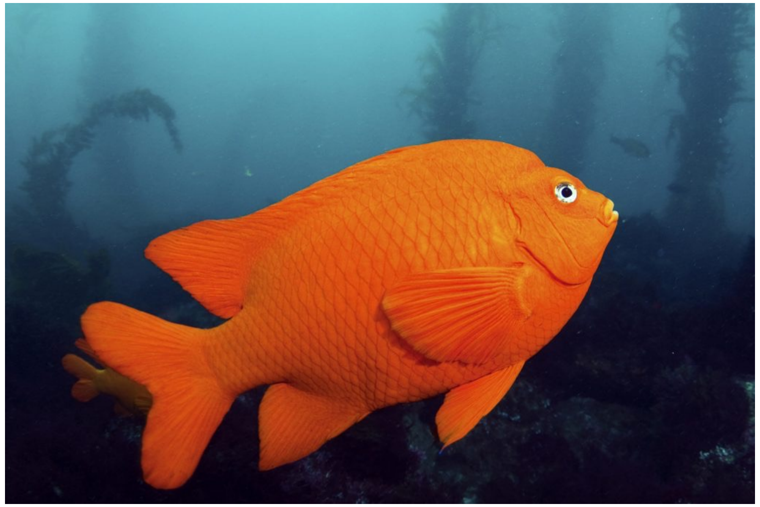
Result follows from boxed theorem. $\blacksquare$

Rest of talk

<u>Lower bounds</u>

Match [Peleg, Shpilka, Volk 22] up to log factor

- $\chi\left(|H\rangle^{\otimes n}\right) \geq \Omega(n/\log n)$.
- There exists $\delta > 0$ such that $\chi_\delta\left(|H\rangle^{\otimes n}\right) \geq \Omega(\sqrt{n}/\log n)$.



<u>Upper bounds</u>

- Generic stabilizer rank

Image: https://fishingbooker.com/blog/california-state-fish-golden-trout-garibaldi/

Upper bounds: Generic stabilizer rank

- Let $\chi_n = \max\{\chi(|\psi\rangle^{\otimes n}) : |\psi\rangle \in \mathbb{C}^2\}$ be the n-th generic stabilizer rank.
- $\chi_n \geq \max\{n+1, \chi(|H\rangle^{\otimes n})\}$ Question: Super-linear lower bound on χ_n?
- Fact: $\chi(|\psi\rangle^{\otimes n}) = \chi_n$ for all but finitely many $|\psi\rangle \in \mathbb{C}^2$ (up to scale).
- Proposition [Lovitz-Steffan]: $\chi_n = O(2^{n/2})$

  (Slight improvement of recent bound $O((n+1)2^{n/2})$ of [Qassim-Pashayan-Gosset 21])
- Fact: There exists a single set of χ_n stabilizer states that can be superimposed to produce any state of the form $|\psi\rangle^{\otimes n}$. Q: Describe such a set?

Summary

Classical simulation of Clifford+T circuits via stabilizer rank

Lower bounds

Match [Peleg, Shpilka, Volk 22] up to log factor

- $\chi\left(|H\rangle^{\otimes n}\right) \geq \Omega(n/\log n)$.
- There exists $\delta > 0$ such that $\chi_\delta\left(|H\rangle^{\otimes n}\right) \geq \Omega(\sqrt{n}/\log n)$.

Upper bounds

- Generic stabilizer rank

https://thumbs.dreamstime.com/b/goldfish-gold-fish-bowl-cute-cartoon-character-happy-145738808.jpg

New techniques for bounding stabilizer rank

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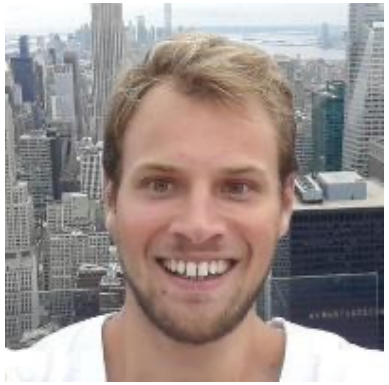

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[arXiv:2110.07781](https://arxiv.org/abs/2110.07781)

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