Project Description

Introduction

The PI uses geometric and combinatorial techniques to study tensor decompositions, with applications in algebraic statistics and quantum information theory. A *tensor* is a multilinear map. These objects naturally generalize matrices, and are ubiquitous in math and science. They can be used to describe nearly any dataset, and any pure (or even, mixed) quantum state. A *decomposition* of a tensor T is an expression of T as a sum of "elementary tensors," which are defined according to the application. For a fixed choice of elementary tensors, the *rank* of T is the minimum number of elementary tensors that can appear in a decomposition of T.

Tensor decompositions are useful in many areas. An important question in quantum information is whether quantum computers can be efficiently simulated by classical computers. This question can be partially resolved by determining the rank of a certain tensor, with elementary tensors given by the set of *stabilizer tensors* [4]. In many other contexts, the elementary tensors are chosen to be the product tensors (i.e. elements of the Segre variety), and the rank of a tensor under this choice is called its *tensor rank*. In quantum physics, product tensors represent unentangled (pure) quantum states, and the tensor rank captures, to some extent, the amount of entanglement present in a state [7]. In complexity theory, the tensor rank of the matrix multiplication tensor precisely quantifies the number of arithmetic operations required for matrix multiplication [5]. In data science, if a tensor represents a dataset, then a tensor rank decomposition can be used to both compress and interpret the data. If the tensor has only one tensor rank decomposition (i.e. the decomposition is *unique*), then there is only one consistent interpretation of the data. For this reason, uniqueness is very useful in data processing [25].

Research objective, methods, and significance

The PI's research objectives are two-fold:

1. Further develop a **matroid theory for product tensors**, and use this to study tensor decompositions.
2. Develop our understanding of **quantum entanglement** as a resource for quantum information processing. In particular:
   (a) Determine the most useful quantum states for distributed information processing tasks.
   (b) Construct and certify entanglement in linear subspaces.
   (c) Classify the invertible maps that can create entanglement.
   (d) Prove bounds on stabilizer rank.

In the remainder of this project description, the PI will describe the significance of each objective, detail his previous work and research methods, and outline his plans for future work. He will then describe how this project will contribute to his career development, justify his choice of sponsoring scientist and host institution, and comment on the broader impacts of this project.

Objective 1: Matroid theory for product tensors

A decomposition of a tensor T into product tensors is completely described by the set of product tensors that appear in the decomposition. Since matroid theory is dedicated to studying (abstractions of) sets of vectors, a natural approach to studying tensor decompositions is to characterize the matroidal structure of sets of product tensors. Despite this, and quite surprisingly, this angle remains largely unexplored. Nevertheless, the PI has found that there is quite a lot of structure, and strong applications to well-known tensor problems.

Problem 1. *Characterize the matroidal structure of sets of product tensors.*

The PI conjectured a "splitting theorem" for sets of product tensors in [28], which he recently proved with Fedor Petrov [31]. In brief, the splitting theorem gives sufficient conditions for a set of product tensors to split (i.e. to be disconnected as a matroid). An immediate corollary to the splitting theorem is a generalization of Kruskal's theorem.

Kruskal's theorem, dating back to 1977, is a famous and widely-used sufficient condition for a tensor rank decomposition to be unique [26]. Until now, the only known extensions of Kruskal's theorem have used Kruskal's original permutation lemma. As a result, these extensions suffer similar drawbacks as Kruskal's theorem does, namely, they require the so-called "k-ranks" to be above a certain threshold. The PI's generalization uses a completely new proof technique, strengthens and unifies many of these extensions, and does not require the k-ranks to be large. Aside from generalizing Kruskal's theorem, the splitting theorem also implies several other powerful results. For example, it implies a lower bound on tensor rank which generalizes Sylvester's matrix rank inequality; and it implies uniqueness-type results for non-rank decompositions (decompositions of a tensor into a non-minimal number of product tensors), which appear to be the first known results of this kind.

Given the strong applications of the splitting theorem, it is natural to ask what more can be said in regards to Problem 1. In the remainder of this section, the splitting theorem is described in more detail, and specific, approachable sub-problems of Problem 1 are identified for future work.

For vector spaces $\mathcal{V}$ and $\mathcal{W}$ over a field $\mathbb{F}$, a *product tensor* is a non-zero tensor of the form $v \otimes w \in \mathcal{V} \otimes \mathcal{W}$. (This definition extends naturally to more than two factors, and many of the statements below also generalize to this setting.) The PI finds it useful to regard a multiset of product tensors $\{v_1 \otimes w_1, \dots, v_n \otimes w_n\}$ as a product $V \circ W$ of matroid representations $V = \{v_1, \dots, v_n\}$ and $W = \{w_1, \dots, w_n\}$, which he calls the *Hadamard product*.

Splitting theorem (Lovitz-Petrov [31])**:** *If* $\operatorname{rank}(V \circ W) \leq \operatorname{rank}(V) + \operatorname{rank}(W) - 2$, *then* $V \circ W$ *splits (i.e. is disconnected as a matroid).*

In the theorem statement, $\operatorname{rank}(\cdot)$ denotes the dimension of the span. An interesting and approachable sub-problem of Problem 1 is to describe the Hadamard product of binary and ternary matroids. In these cases, it can be shown that the Hadamard product is a well-defined product of *matroids*, and not just of representations. The splitting theorem already provides a step toward resolving this sub-problem. If the splitting theorem is any indication, further progress on this sub-problem could have strong implications for tensor decompositions over $\mathbb{F}_2$ and $\mathbb{F}_3$. Tensor rank over $\mathbb{F}_2$ has deep connections to algebraic complexity theory and coding theory [5, Chapter 18].

As further angles on Problem 1, the PI is developing a robust version of the splitting theorem with Aravindan Vijayaraghavan, which would in particular imply a generalization of a recent robust version of Kruskal's theorem [3]. The PI is also collaborating with Luca Chiantini, Fedor Petrov, and Pierpaola Santarsiero to use the splitting theorem and previous ideas developed for Waring rank decompositions to find more uniqueness results.

The PI first came to conjecture the splitting theorem while studying the following question, which he introduced in [29]: Given a set of unit vectors $\{v_1, \dots, v_n\} \subseteq \mathbb{C}^d$, when does there exist a positive integer m and an isometry $U : \mathbb{C}^d \to (\mathbb{C}^2)^{\otimes m}$ such that Uv_a is a product tensor for all $a \in [n]$? The PI's question has become a topic of recent interest in quantum information [6, 39, 27].

Objective 2: Develop our understanding of quantum entanglement as a resource

A *pure (quantum) state* is a normalized tensor in a tensor product of $\mathbb{C}$-vector spaces, modulo phase. More generally, a *(quantum) state* is a probabilistic mixture of pure quantum states, or equivalently, a positive semi-definite operator of unit trace [37].

In order to build a quantum computer, and further our understanding of the physical universe, it is paramount to better understand quantum entanglement as a resource [35, 21]. To this end, the PI will pursue the following objectives.

Objective 2(a): Determine the most useful quantum states for distributed tasks

It is practical to ask which entangled states are the most useful for non-local quantum information processing. A natural non-local setting occurs when multiple spatially separated parties wish to jointly execute some task, but can only perform quantum operations within their own laboratories. Assuming that the parties are allowed to use pre-shared entanglement to help them perform the task, what are the most useful entangled states for them to share?

Problem 2. *What are the most useful entangled states for non-local quantum information processing?*

In the two-party setting, we know that the most useful state is (the local unitary orbit of) the *canonical maximally entangled state*, as it can be locally converted into any other two-way state [34]. However, in the multi-party setting this question remains unanswered. A natural sub-problem is to determine the most useful states for specific, fundamental tasks.

In recent work, the PI studied with Nathaniel Johnston the task of local state discrimination [30]. For a given entangled pure state T, the number of generic pure states that can be locally discriminated with pre-shared entanglement T was computed. It was found that this number is precisely the Krull dimension of the orbit closure of T under the action of the product general linear group. In quantum information circles, this orbit is known as the *SLOCC orbit* of T. An immediate, yet surprising consequence is that a Zariski-open-dense (full-measure) set of states T maximize this dimension, and hence are maximally useful for local state discrimination.

As further steps toward resolving Problem 2, the PI intends to determine the most useful states for other fundamental multi-party tasks, such as multi-party quantum key distribution [15] and the multi-party quantum XOR game [38].

Objective 2(b): Construct and certify entanglement in linear subspaces

We say that a linear subspace of matrices is (ϵ, r)*-entangled* if every pure state in that subspace has distance greater than ϵ from the set of pure states of rank $\leq r$. Entangled subspaces have applications in quantum error correction [17, 36], quantum tomography [18], entanglement witnesses [1, 23, 24], and one of the most important open problems in quantum information [20]:

Problem 3. *For a positive integer r, do there exist quantum states that are non-positive under partial transposition (NPT) and are not r-distillable?*

A quantum state ρ is called *r-distillable* if $\rho^{\otimes r}$ can be locally transformed into a maximally entangled (two-way) state, and ρ is called *distillable* if $\rho^{\otimes r}$ can be locally transformed into a non-vanishing number of maximally entangled states as $r \to \infty$. It is known that there exist NPT states that are not 1-distillable, but Problem 3 remains open for $r = 2$.

By a clever reduction due to Michał and Paweł Horodecki, Problem 3 is equivalent to asking whether a certain linear subspace of matrices is $(\epsilon, 2)$-entangled, for a certain constant ϵ [19]. In ongoing work with Nathaniel Johnston and Aravindan Vijayaraghavan, the PI has developed a technique to efficiently certify that a linear subspace is (ϵ, r)-entangled, by considering a certain linear relaxation of the defining system of polynomial equations. The PI is optimistic that this will be a fruitful step forward in approaching Problem 3.

In related work with Nathaniel Johnston, the PI has obtained new explicit constructions of entangled subspaces [30], which can be used to construct *entanglement witnesses*, quantum measurements which certify entanglement [1, 23, 24]. Also related, with Nathaniel Johnston and Daniel Puzzouli, the PI has introduced a method to quantify "how good" a positive linear map is for detecting entanglement [24].

Objective 2(c): Classify the invertible maps that can create entanglement

The *border rank* of a pure quantum state T is the smallest integer r for which T can be approximated arbitrarily well by tensors of rank at most r. The border rank naturally quantifies the amount of entanglement present in T [7]. It is natural to ask which quantum operations can increase entanglement [16]. We say that an invertible linear map is *r-entangling* if it can increase the border rank of a border-rank-r pure state.

Problem 4. *Describe the r-entangling invertible maps.*

It is known that every operator that is not a composition of local operators and swaps is 1-entangling [22]. In ongoing work, the PI has developed a technique to study r-entanglers using *secant multidrop lines*, revealing a connection between this problem and recent studies of border rank sub-multiplicativity [8, 2]. The PI has used this technique to completely characterize the 2-entanglers, and uncovered the first examples of non-entanglers that are non-local and non-swap. In collaboration with William Slofstra and Fulvio Gesmundo, the PI has developed a technique to computationally determine the r-entanglers using Lie theory. In collaboration with William Slofstra, the PI is mentoring an undergraduate student, Daniel Han, with the goal of implementing this computation in Sage and resolving Problem 4 for $r \geq 3$.

Objective 2(d): Prove bounds on stabilizer rank

Many resources are being dedicated to building a quantum computer. It is therefore important to make sure that quantum computers could indeed outperform classical computers.

Problem 5. *Can quantum computers be efficiently simulated by classical computers?*

In similar spirit to how the computational cost of matrix multiplication can be reduced to determining the tensor rank of the matrix multiplication tensor, Problem 5 is closely related to determining the *stabilizer rank* of n-qubit quantum states. A polynomial upper bound on the stabilizer rank of a certain class of states would imply a positive answer to Problem 5, whereas an exponential lower bound would imply a negative answer for state-of-the-art classical simulation protocols [4].

In recent work with Vincent Steffan, the PI has introduced techniques from algebraic geometry and number theory to bound the stabilizer rank [32]. In particular, he has refined a number-theoretic theorem of Moulton to exhibit an explicit sequence of quantum states with exponential stabilizer rank but constant approximate stabilizer rank, and to provide alternate (and simplified) proofs of the best-known asymptotic lower bounds on stabilizer rank and approximate stabilizer rank, up to a log factor. He has also uncovered the first non-trivial examples of quantum states with multiplicative stabilizer rank under the tensor product. Finally, he has used algebraic-geometric techniques to prove new bounds on the generic stabilizer rank.

Contribution to career development

This project will enable the PI to develop his career in academia and collaborate with researchers at the host institution. The NSF support will allow the PI to dedicate more time towards research than would be possible under the heavy teaching load of other postdoctoral appointments. The PI's intended research will help him establish himself in his field, and the support for travel will enable him to continue to share his research at conferences and workshops. The PI will also receive valuable mentorship from the sponsoring scientist, Harm Derksen.

Justification of sponsoring scientist and host institution

The PI will perform the activities outlined in this proposal at Northeastern University under the supervision of the sponsoring scientist, Harm Derksen. Northeastern University has several distinguished faculty members with whom the PI would value the opportunity to interact. In particular, the research of Paul Hand, Christopher King, Valerio Toledano Laredo, and Gabor Lippner in machine learning, quantum information theory, representation theory, and quantum walks align closely with the PI's interests. The Geometry, Physics, and Representation Theory seminar as well as the CS Theory seminar at Northeastern University attract researchers with interests relevant to the PI.

The sponsoring scientist has strong expertise in quiver representations, applied algebraic geometry, and representation theory, which he has used extensively to study tensor decompositions [12, 11, 14, 13]. The PI expects to benefit greatly from the sponsoring scientist's knowledge of these fields, as they are quite relevant to his research objectives. Indeed, the PI has already found applications of algebraic geometry and representation theory in Objectives 2(a)-(d). In Objective 2(d), the set of tensors of fixed stabilizer rank forms a subspace arrangement, a special kind of quiver representation that the sponsoring scientist has extensively studied [9, 10, 33]. For Objective 1, since a matroid representation is a special kind of quiver representation, the PI and sponsoring scientist have discussed a possible extension of the splitting theorem to quiver representations.

Broader impacts

During this project, the PI will undertake several initiatives to counteract the gender and socioeconomic disparity that permeates our STEM community. The PI will mentor high school students from Boston's underserved communities by participating in the "Bridge to Calculus" program. This summer program, developed collaboratively between Northeastern University and the Boston public school system, helps students prepare for college-level math. The PI will also serve as a mentor in the 501(c) non-profit "Science Club for Girls" program, held in Cambridge, which provides free STEM training for K-12 girls on Saturdays.

The PI will also pursue opportunities to mentor student research. In particular, the PI intends to supervise undergraduate students through Northeastern University's MATH 4020, a research capstone project for juniors and seniors.

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