A generalization of Kruskal's theorem

What happens when you replace the Kruskal ranks with standard ranks in Kruskal's theorem?

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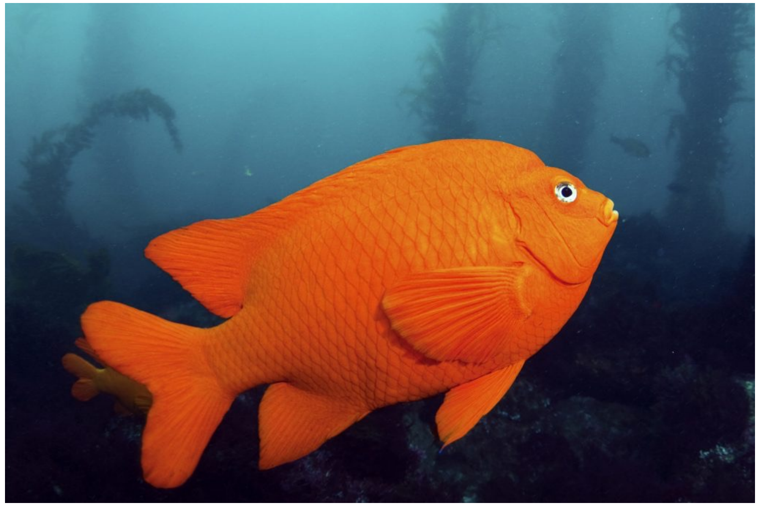
Slides available at benjaminlovitz.com

Outline

- Uniqueness of tensor decompositions
- Kruskal's theorem
- A generalization of Kruskal's theorem
- Our main result



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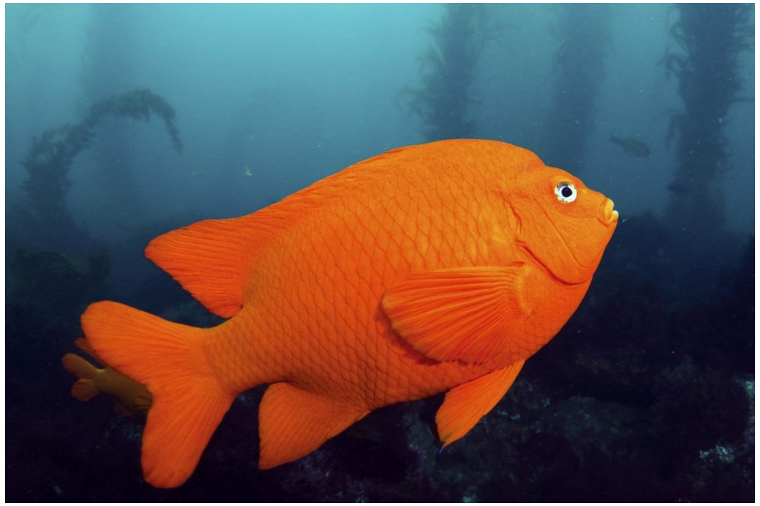

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Uniqueness of tensor decompositions

<u>Definition:</u> Let $n \in \mathbb{N}$ and $[n] := \{1, \dots, n\}$.

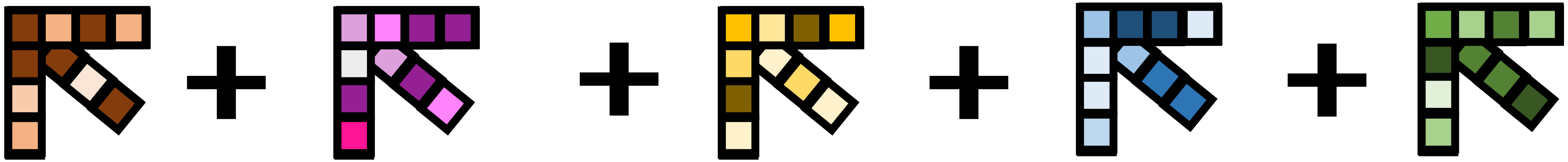
A tensor decomposition

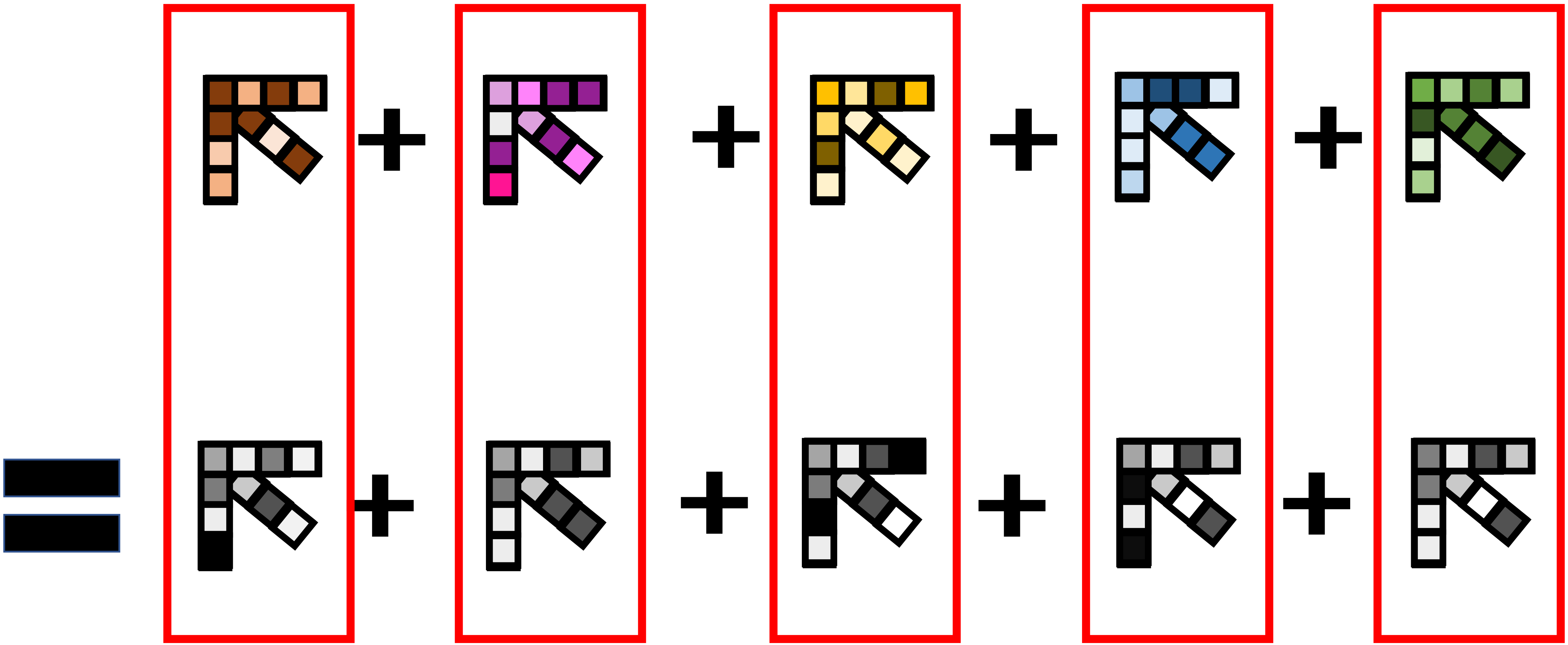
$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z$$

is called the unique decomposition of T if for any other decomposition

$$T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a \in X \otimes Y \otimes Z$$

there is a permutation $\sigma \in S_n$ such that $x_a \otimes y_a \otimes z_a = x'_{\sigma(a)} \otimes y'_{\sigma(a)} \otimes z'_{\sigma(a)}$ for all $a \in [n]$. $\Rightarrow \text{tensor rank}(T) = n$

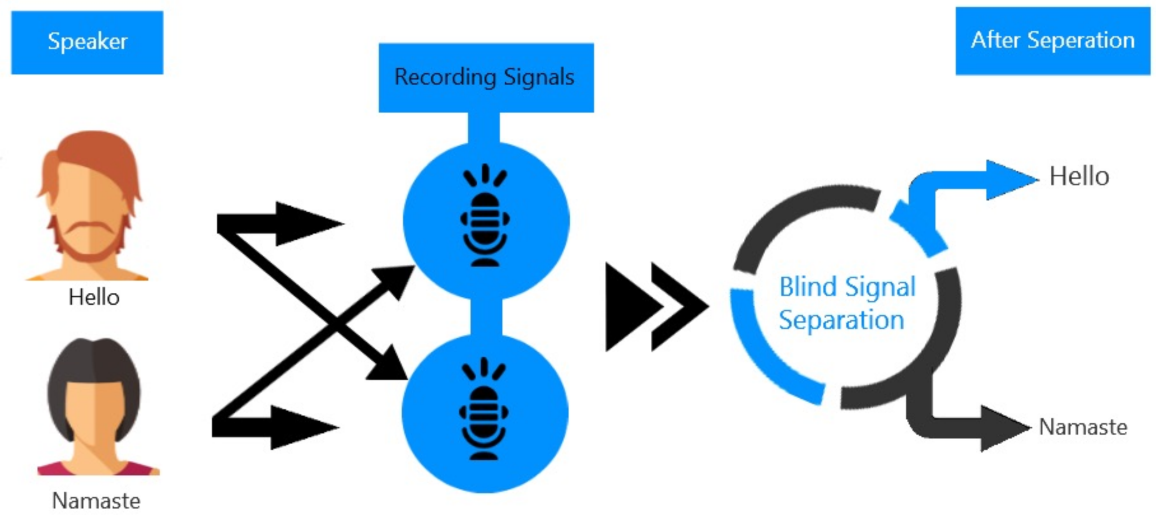




Applications

- Tensors ⟷ Physical data
- Tensor decomposition ⟷ Interpretation of data
- Unique decomposition ⟷ Unique interpretation

Blind Signal Separation



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Goal: Separate mixed signals + determine mixing process

Method: Decompose tensor arising from measurement data

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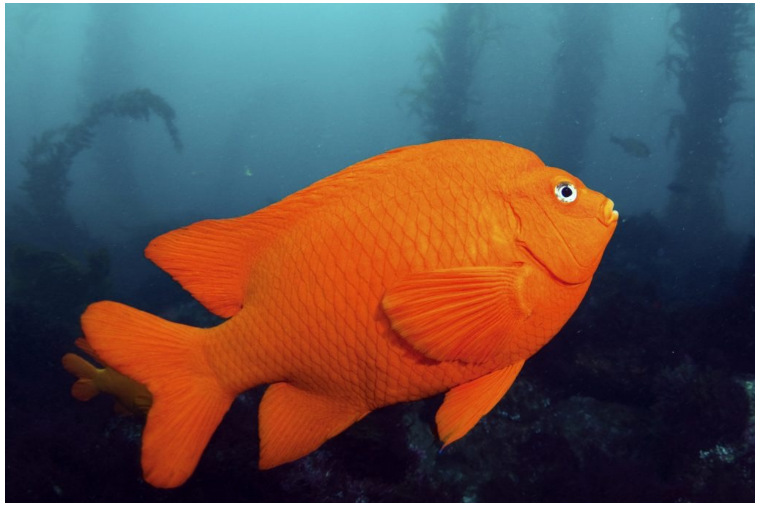

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Kruskal's theorem

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

Notation: When $S \subseteq [n]$, $d_x^S := \mathrm{dimspan}\{x_a : a \in S\}$

$$d_y^S := \mathrm{dimspan}\{y_a : a \in S\}$$

$$d_z^S := \mathrm{dimspan}\{z_a : a \in S\}$$

Kruskal's theorem

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Example:

$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} + e_4^{\otimes 3} + (e_1 + e_2) \otimes (e_2 + e_3) \otimes (e_1 + e_4)$$

$x_1 \otimes y_1 \otimes z_1$ … x_5 $\otimes$ y_5 $\otimes$ z_5

For $S = \{1,2,5\}$, $\quad d_x^S = 2,\ d_y^S = 3,\ d_z^S = 3$

Kruskal's theorem

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Definition: The Kruskal rank of $\{x_1, \ldots, x_n\} \in X$ is the largest integer k_x such that for every subset $S \subseteq [n]$ of size $|S| = k_x$, it holds that

$d_x^S = |S|$.

$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} + e_4^{\otimes 3} + (e_1 + e_2) \otimes (e_2 + e_3) \otimes (e_1 + e_4)$$

$x_1 \otimes y_1 \otimes z_1$ … x_5 $\otimes$ y_5 $\otimes$ z_5

$$\{x_1, \ldots, x_5\} = \{e_1, e_2, e_3, e_4, e_1 + e_2\}, \qquad k_x = 2,\ d_x^{[5]} = 4.$$

Kruskal's theorem

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

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Kruskal's theorem: If $2n \leq k_x + k_y + k_z - 2$, then (1) is the unique decomposition of T.

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Kruskal's theorem: If $2n \leq k_x + k_y + k_z - 2$, then (1) is the unique decomposition of T.

Example [Jennrich's Theorem]:

$\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ are linearly independent, and $k_z \geq 2$.

$k_x = k_y = n$

Kruskal’s theorem

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

Kruskal’s theorem: If $2n \leq k_x + k_y + k_z - 2$, then (1) is the unique decomposition of T.

Weakness: If T is unique under Kruskal’s theorem, then so is

$$T' = \sum_{a \in [n]} x_{\sigma(a)} \otimes y_a \otimes z_a \in X \otimes Y \otimes Z$$

for any $\sigma \in S_n$.

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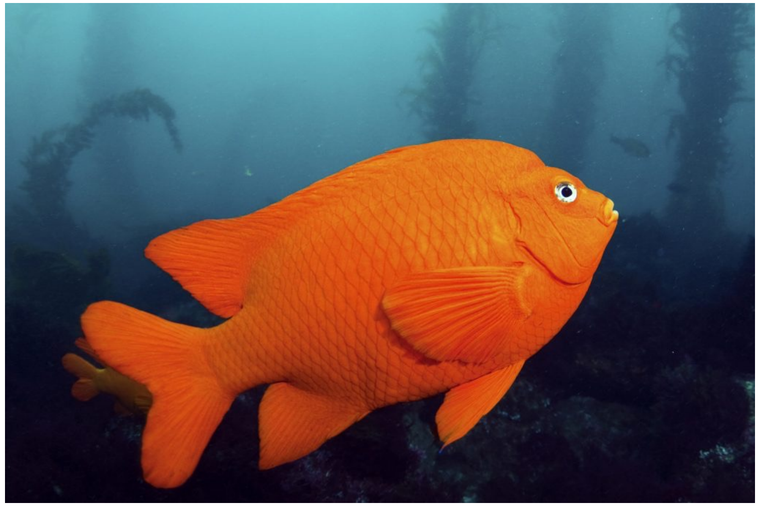

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Our generalization

$$T = \sum_{a\in[n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

Idea: Replace Kruskal ranks k_x with standard dimspans $d_x^{[n]}$.

<u>Theorem [L-Petrov]:</u> If

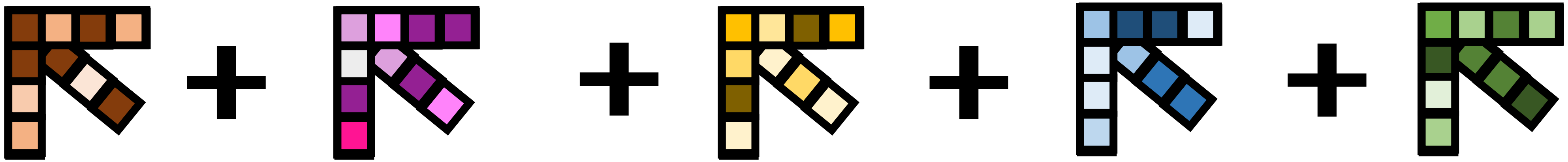
$d_x^{[n]} = \text{dimspan}\{x_1, \ldots, x_n\}$

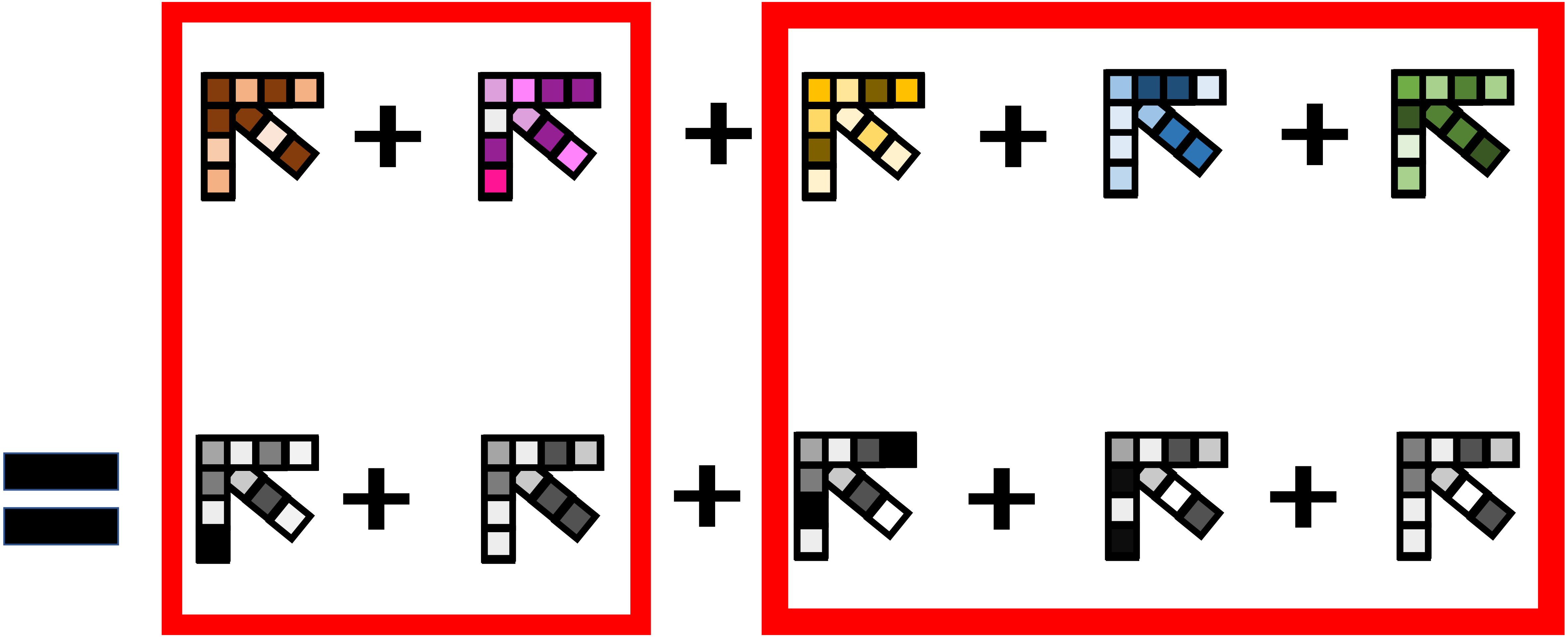
$$2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then for any other decomposition $T = \sum_{a\in[n]} x'_a \otimes y'_a \otimes z'_a \in X \otimes Y \otimes Z$

there exist non-trivial subsets $S, T \subseteq [n]$ such that

$$\sum_{a\in S} x_a \otimes y_a \otimes z_a = \sum_{a\in T} x'_a \otimes y'_a \otimes z'_a$$





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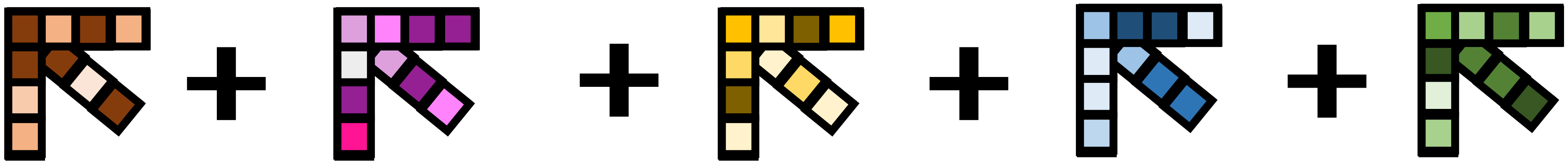
Idea: Replace Kruskal ranks k_x with standard dimspans d_x^S.

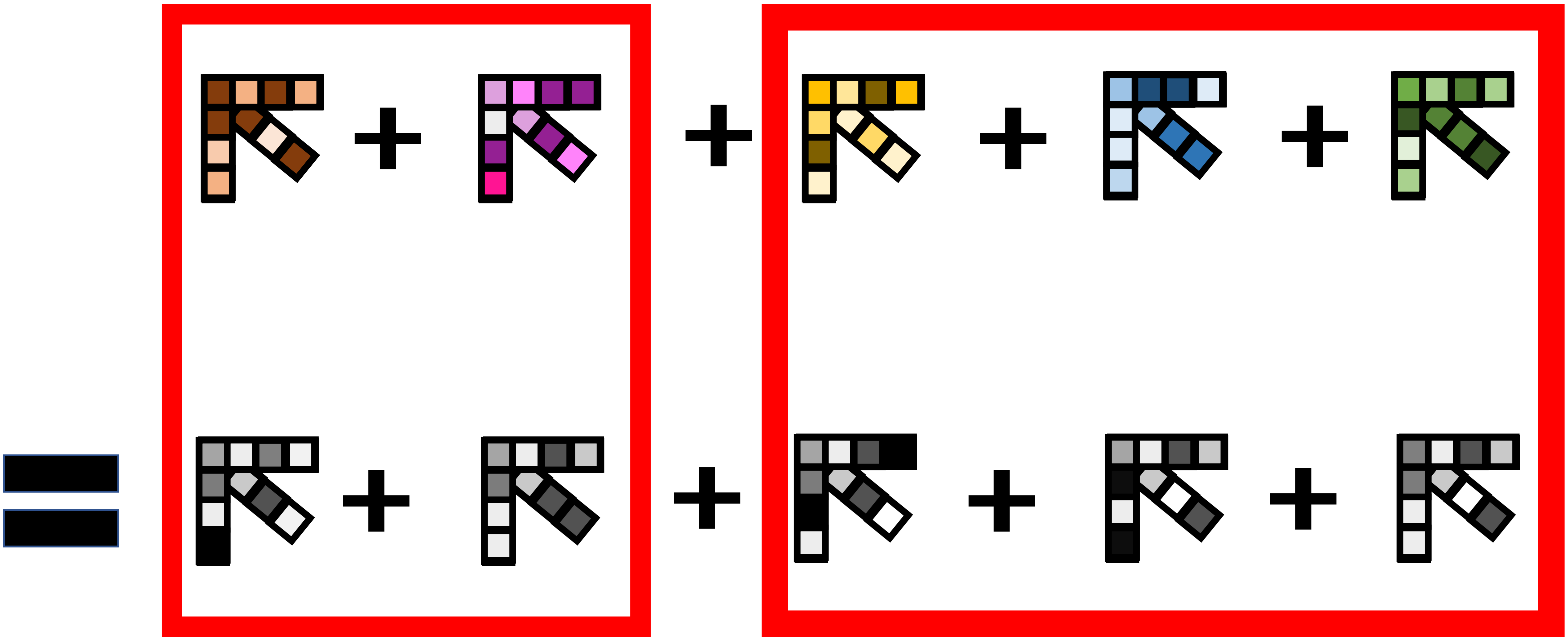
<u>Theorem [L-Petrov]:</u> If for every subset $S \subseteq [n]$ of size $|S| \geq 2$, it holds that

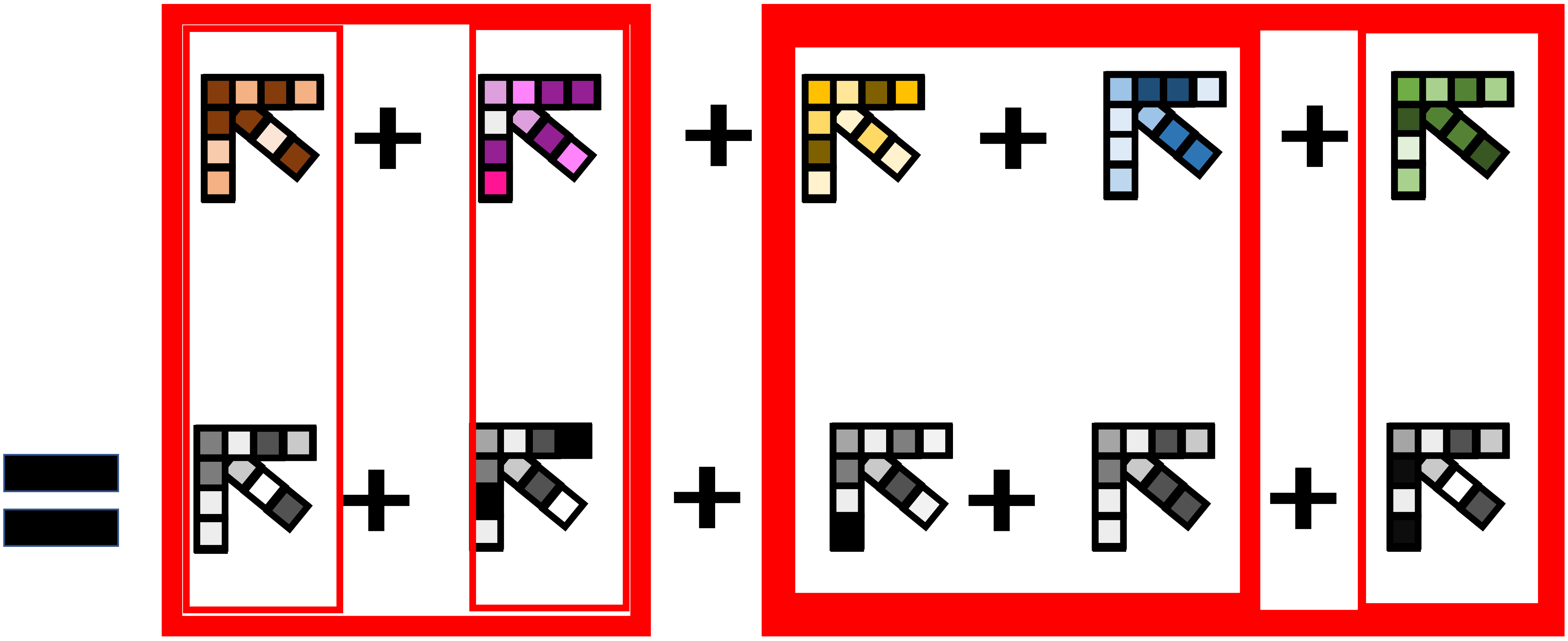
$$2|S| \leq d_x^S + d_y^S + d_z^S - 2,$$

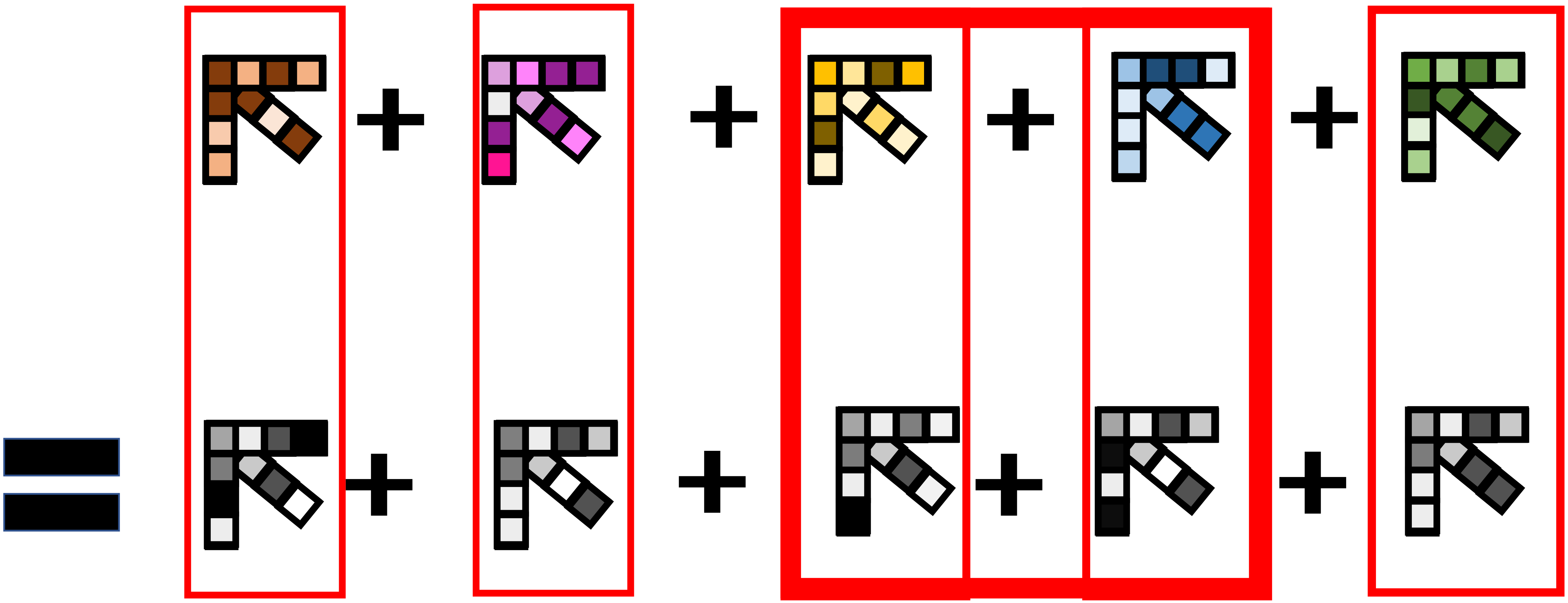
$d_x^S = \mathrm{dimspan}\{x_a : a \in S\}$

then (1) is the unique decomposition of T.









Our generalization

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

Idea: Replace Kruskal ranks k_x with standard dimspans d_x^S.

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Example

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} + e_4^{\otimes 3} + (e_1 + e_2) \otimes (e_2 + e_3) \otimes (e_1 + e_4)$$

$x_1 \otimes y_1 \otimes z_1$ … x_5 $\otimes$ y_5 $\otimes$ z_5

Kruskal's theorem does not certify uniqueness

$$10 = 2n \nleq k_x + k_y + k_z - 2 = 2 + 2 + 2 - 2 = 4$$

Example

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

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$x_1 \otimes y_1 \otimes z_1$ … x_5 $\otimes$ y_5 $\otimes$ z_5

Theorem [L-Petrov]: If for every subset $S \subseteq [n]$ of size $|S| \geq 2$, it holds that $2|S| \leq d_x^S + d_y^S + d_z^S - 2$,

then (1) is the unique decomposition of T.

$10 = 2|[n]| \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2 = 4 + 4 + 4 - 2 = 10$ ✓

For $S = \{1,2\}$, $\quad 4 = 2|S| \leq d_x^S + d_y^S + d_z^S - 2 = 2 + 2 + 2 - 2 = 4$ ✓

For $S = \{1,2,5\}$, $\quad 6 = 2|S| \leq d_x^S + d_y^S + d_z^S - 2 = 2 + 3 + 3 - 2 = 6$ ✓

For $S = \{1,2,3,5\}$, $\quad 8 = 2|S| \leq d_x^S + d_y^S + d_z^S - 2 = 3 + 3 + 4 - 2 = 8$ ✓

By symmetric arguments for other $S \subseteq [n]$, this is the unique decomposition of T

Example

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

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$x_1 \otimes y_1 \otimes z_1$ … x_5 $\otimes$ y_5 $\otimes$ z_5

Unique by [L-Petrov], but

$$T' = \sum_{a \in [5]} x_{\sigma(a)} \otimes y_a \otimes z_a$$

not unique for $\sigma = (13) \in S_5$

A few other uniqueness results

- Two batches I have read:
  1. [Domanov-Sørensen-De Lathauwer, D-De L, S-De L],
     - Kruskal's proof on steroids
     - Requires Kruskal ranks to be above a certain threshold

  2. [Ballico-Bernardi-Chiantini-Guardo, C-Sacchi]
     - Our main result (coming next!) reproduces key lemmas

Related: Uniqueness of symmetric decompositions, generic uniqueness, uniqueness for low rank tensors, finite decompositions,…

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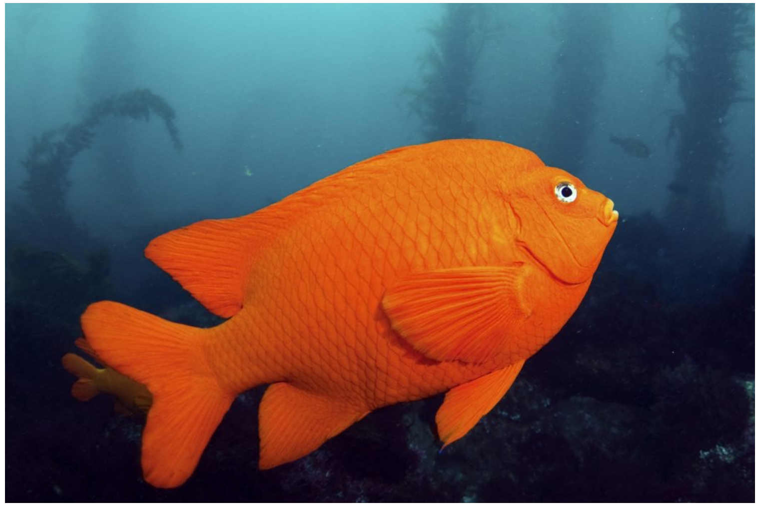

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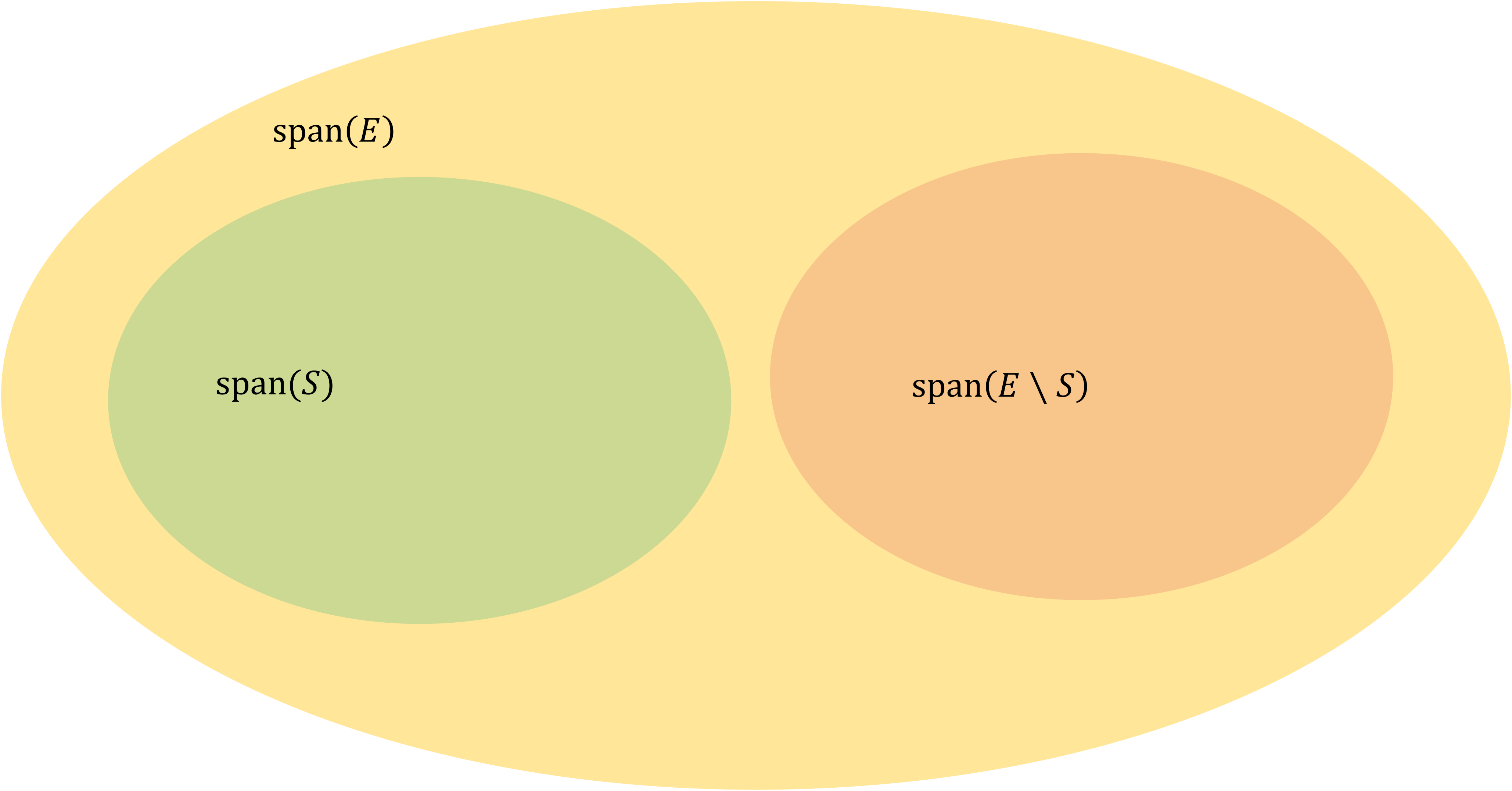
Splitting

Definition: A set of vectors $E = \{v_1, \dots, v_n\}$ splits if there exists a non-trivial subset $S \subseteq E$ such that

$$\mathrm{span}(E) = \mathrm{span}(S) \oplus \mathrm{span}(E \setminus S). \tag{2}$$

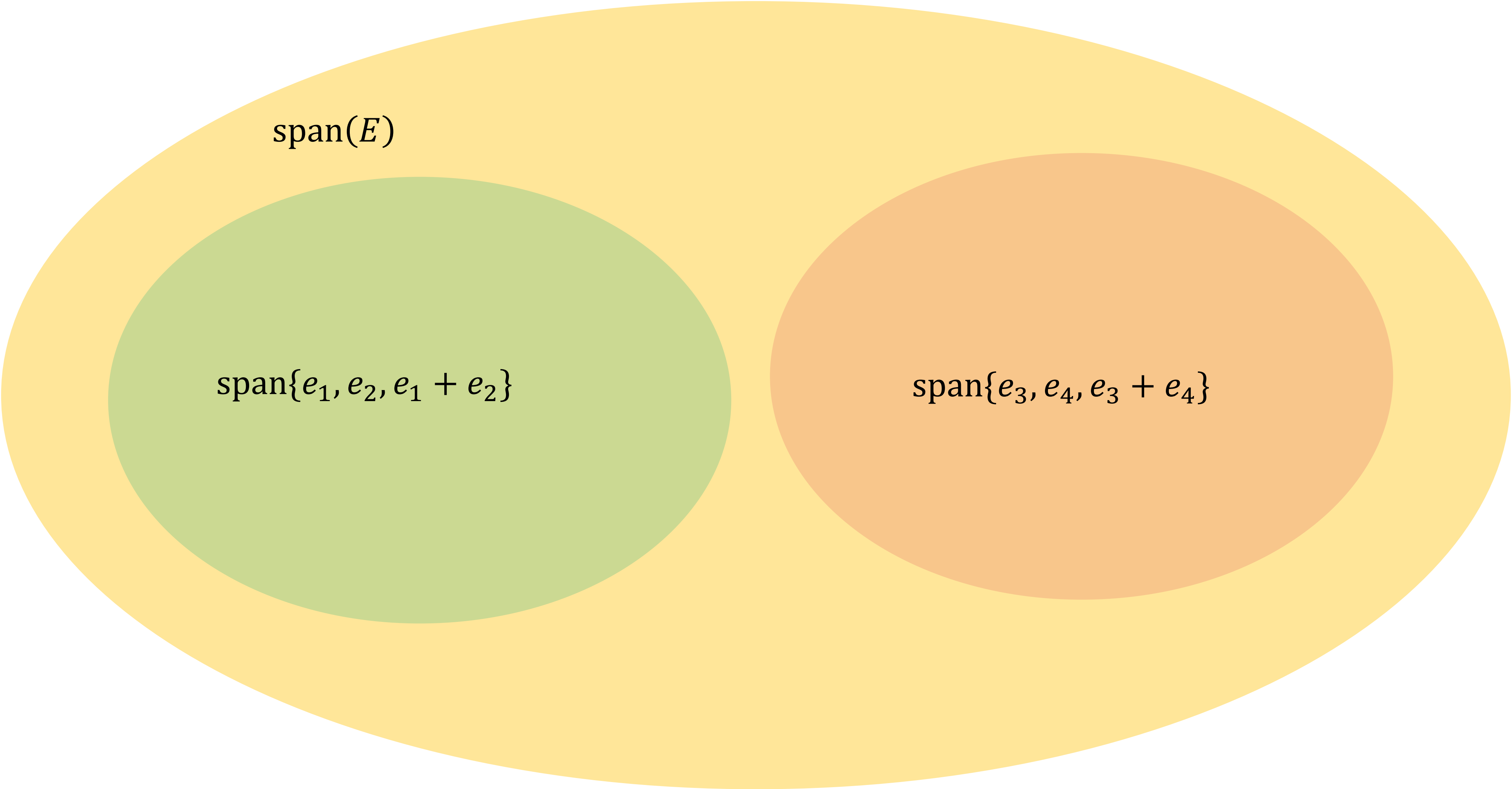
$$\Leftrightarrow \mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$$

E splits if there exists $S \subseteq E$ such that
$\mathrm{span}(E) = \mathrm{span}(S) \oplus \mathrm{span}(E \setminus S)$



E splits if there exists $S \subseteq E$ such that $\text{span}(E) = \text{span}(S) \bigoplus \text{span}(E \setminus S)$

$E = \{e_1, e_2, e_1 + e_2, e_3, e_4, e_3 + e_4\}$



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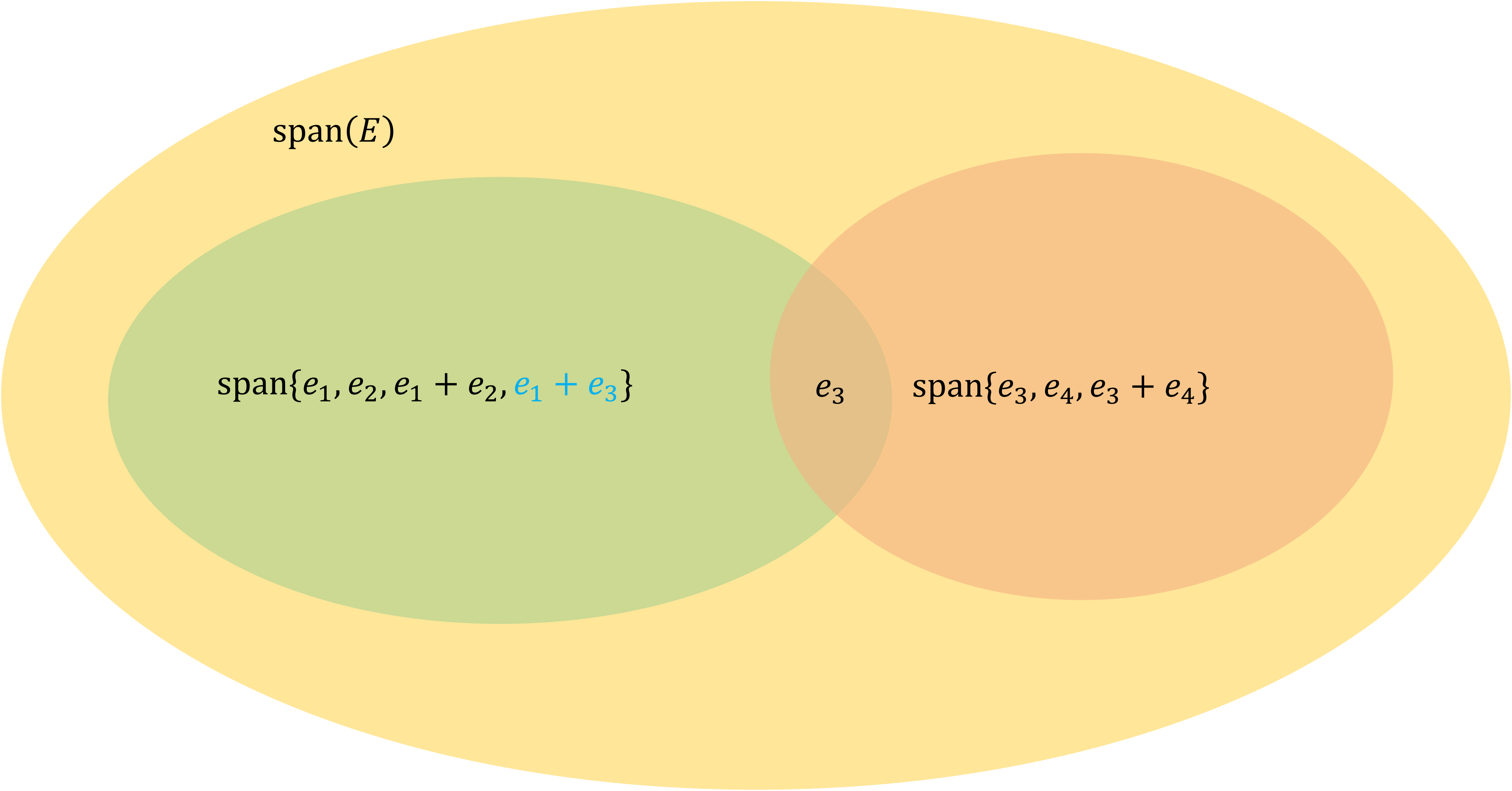
$E = \{e_1, e_2, e_1 + e_2, e_3, e_4, e_3 + e_4\} \cup \{e_1 + e_3\}$



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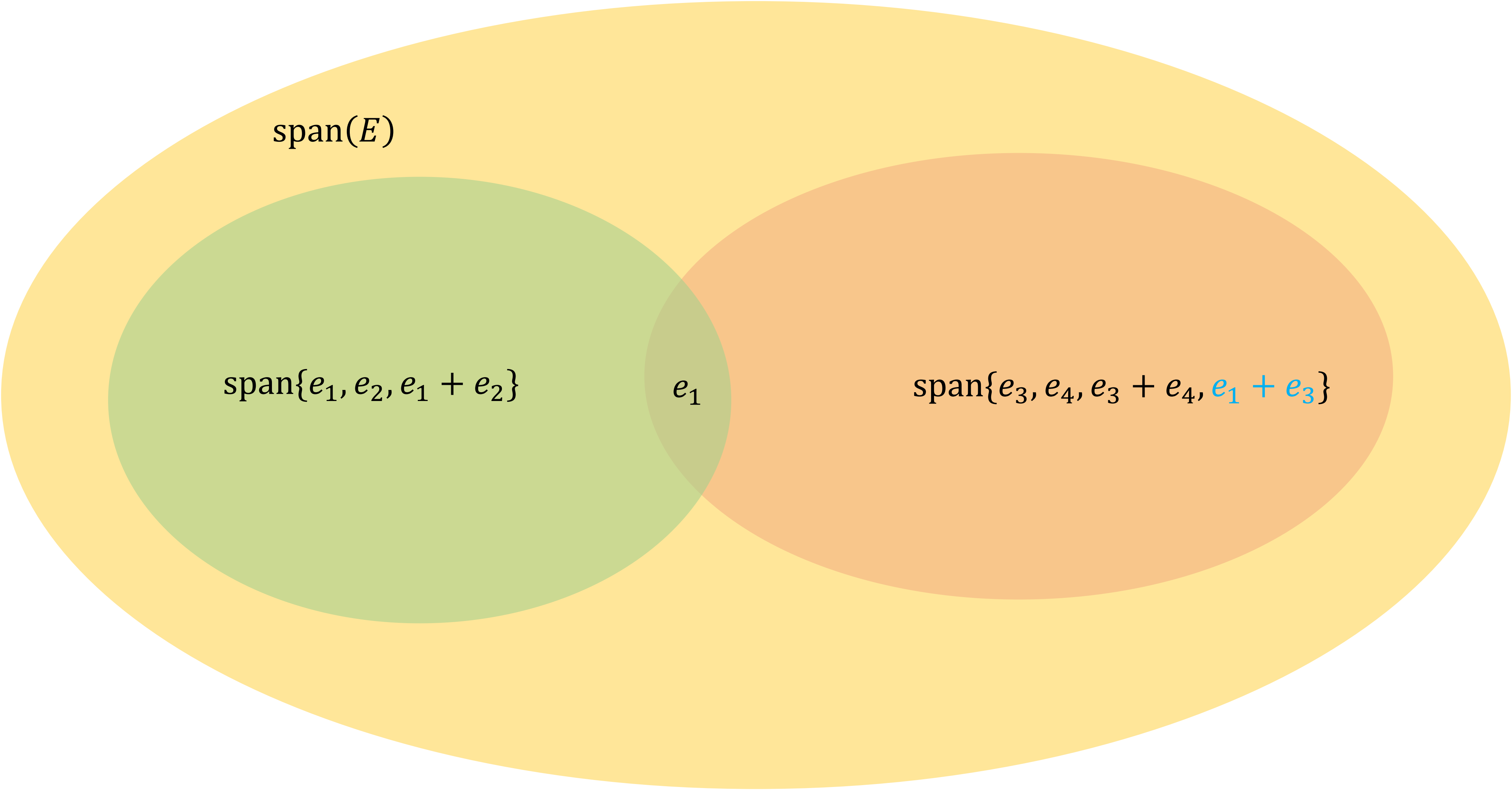

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Fact: If (2) holds, and $\sum(E) = 0$, then $\sum(S) = \sum(E \setminus S) = 0$

Proof: $\sum(E) = 0 \Rightarrow \sum(S) = -\sum(E \setminus S) \in \mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$ ■

Main result

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(E) = \mathrm{span}(S) \oplus \mathrm{span}(E \setminus S)$

Main result [L-Petrov]: Let $E = \{x_a \otimes y_a : a \in [n]\}$.

If

$$\mathrm{dimspan}(E) \leq d_x^{[n]} + d_y^{[n]} - 2$$

then E splits.

$d_x^{[n]} = \mathrm{dimspan}\{x_1, \dots, x_n\}$

(Our generalization of Kruskal's theorem is a corollary to this)

Main result

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(E) = \mathrm{span}(S) \oplus \mathrm{span}(E \setminus S)$

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$d_x^{[n]} = \mathrm{dimspan}\{x_1, \dots, x_n\}$

Corollary: If

$$n \leq d_x^{[n]} + d_y^{[n]} - 1,$$

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Main result

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Corollary: If

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Sylvester's rank inequality: $n \leq d_x^{[n]} + d_y^{[n]} - \mathrm{rank}(XY^T)$

Main result

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Corollary: If

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then E splits.

Example: $E = \{e_1 \otimes e_1, \quad e_2 \otimes e_2, \quad e_3 \otimes e_2, \quad (e_2+e_3) \otimes e_2\}$ splits.

$$4 = n \leq d_x^{[4]} + d_y^{[4]} - 1 = 3 + 2 - 1 = 4$$

Main result

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(E) = \mathrm{span}(S) \oplus \mathrm{span}(E \setminus S)$

Main result [L-Petrov]: Let $E = \{x_a \otimes y_a : a \in [n]\}$.

If

$$\mathrm{dimspan}(E) \leq d_x^{[n]} + d_y^{[n]} - 2$$

then E splits.

$d_x^{[n]} = \mathrm{dimspan}\{x_1, \ldots, x_n\}$

Corollary: If

$$n \leq d_x^{[n]} + d_y^{[n]} - 1,$$

then E splits.

Main result => Corollary: If E is linearly independent, then it splits.

Otherwise, $\mathrm{dimspan}(E) \leq n - 1 \leq d_x^{[n]} + d_y^{[n]} - 2,$ so E splits by main result.

Main result

E splits if there exists $S \subseteq E$ such that $\mathrm{span}(E) = \mathrm{span}(S) \oplus \mathrm{span}(E \setminus S)$

Corollary [L-Petrov]: Let $E = \{x_a \otimes y_a \otimes z_a : a \in [n]\}$.

If

$$\dim\mathrm{span}(E) \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 3$$

then E splits.

$d_x^{[n]} = \dim\mathrm{span}\{x_1, \ldots, x_n\}$

Corollary: If

$$n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then E splits.

Main result

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Corollary [L-Petrov]: Let $E = \{x_a \otimes y_a \otimes z_a : a \in [n]\}$.

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then E splits.

Corollary: If

$$n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then E splits.

Corollary: If

$$2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then for any other set of product tensors $E' = \{x_a' \otimes y_a' \otimes z_a' : a \in [n]\}$, $E \cup E'$ splits.

Corollary => Kruskal generalization

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

Suffices to prove:

<u>Theorem [L-Petrov]:</u> If

$d_x^{[n]} = \text{dimspan}\{x_1, \ldots, x_n\}$

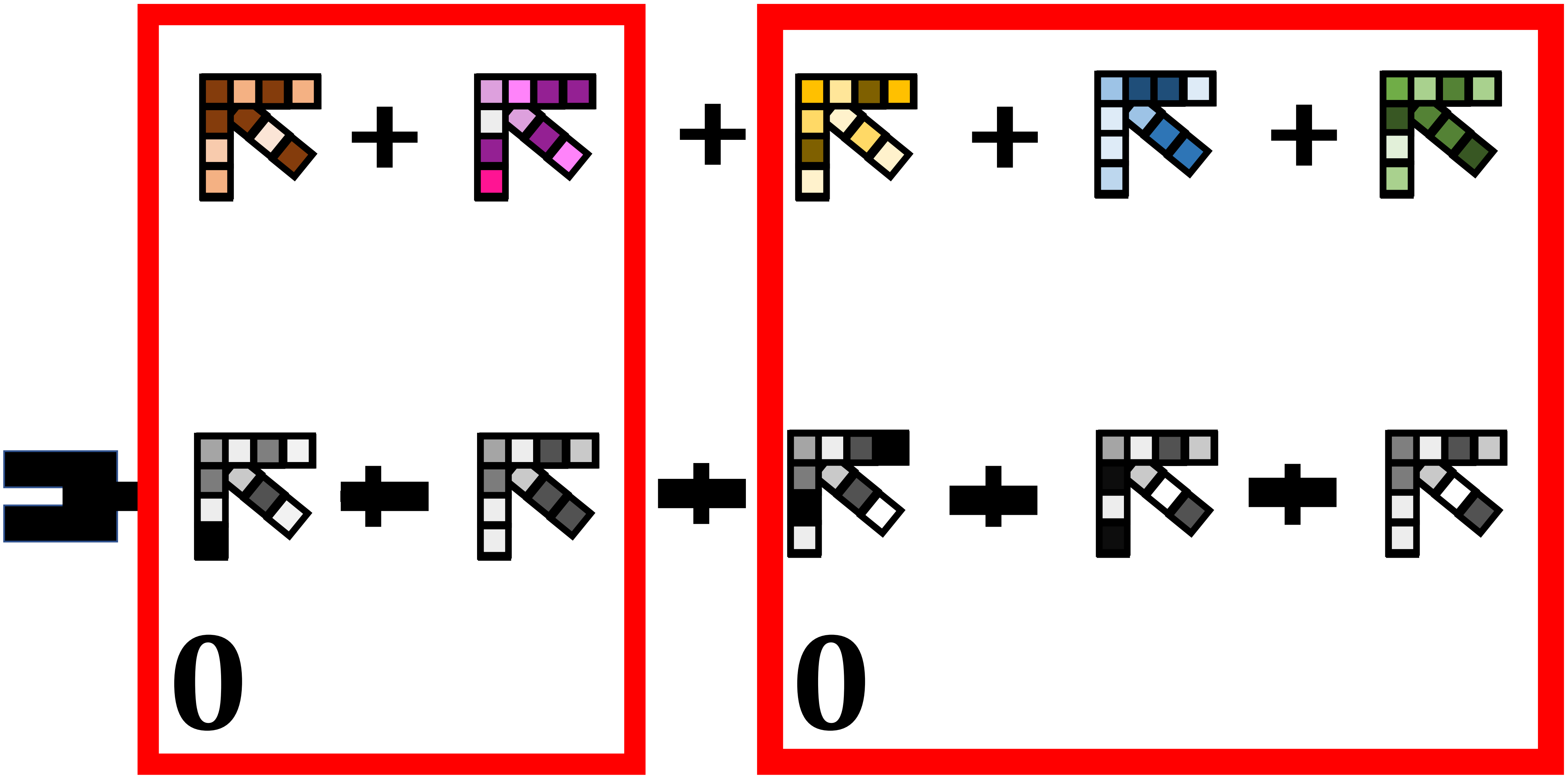
$$2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then for any other decomposition $T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a \in X \otimes Y \otimes Z$

there exist non-trivial subsets $S, T \subseteq [n]$ such that $\sum_{a \in S} x_a \otimes y_a \otimes z_a = \sum_{a \in T} x'_a \otimes y'_a \otimes z'_a$

<u>Proof:</u>

By previous corollary, $\{x_a \otimes y_a \otimes z_a, x'_a \otimes y'_a \otimes z'_a \ : a \in [n]\}$ splits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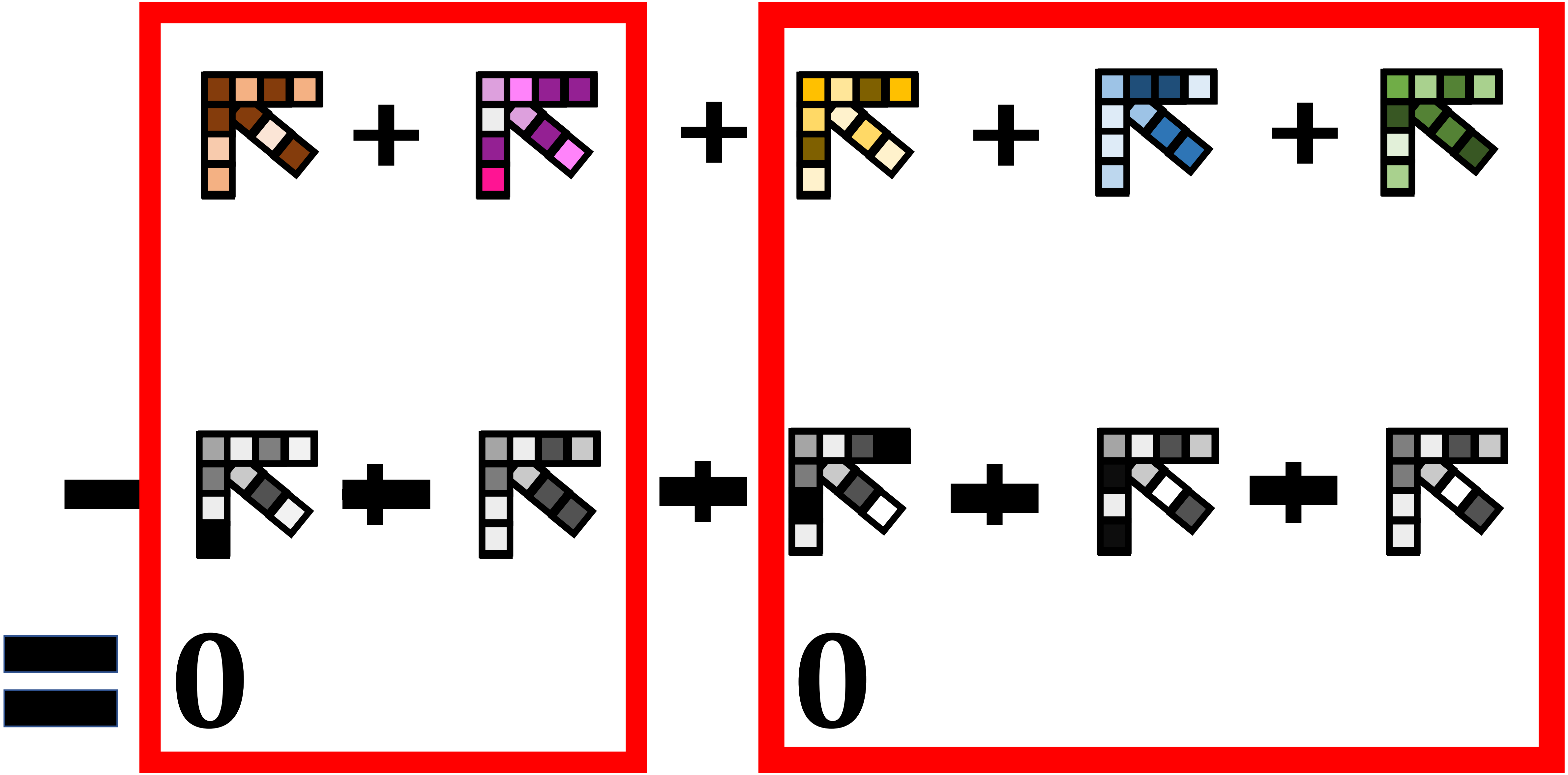
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Conclusion

- Uniqueness of tensor decompositions
- Kruskal's theorem
- A generalization of Kruskal's theorem
- Our main result
- What else can we do with the main result?

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