A generalization of Kruskal’s theorem

What happens when you replace the Kruskal ranks with standard ranks in Kruskal’s theorem?

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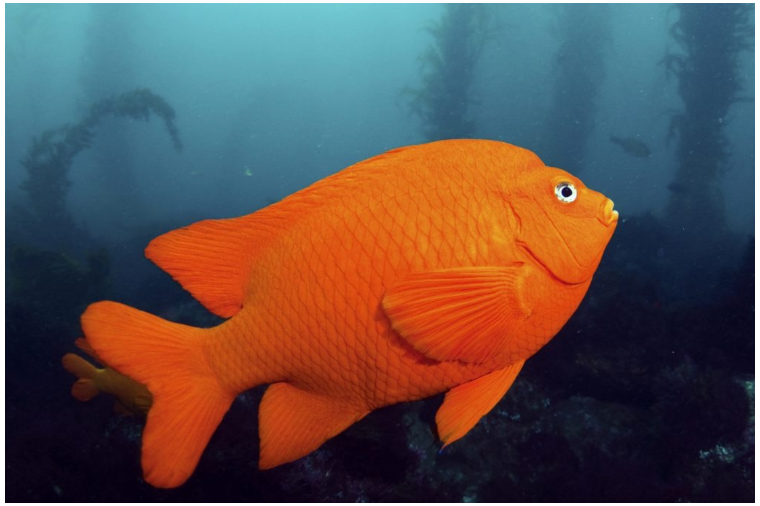
Slides available at www.benjaminlovitz.com

Outline

- Uniqueness of tensor decompositions
- Kruskal's theorem
- A generalization of Kruskal's theorem
- Splitting theorem
- Another generalization of Kruskal's theorem



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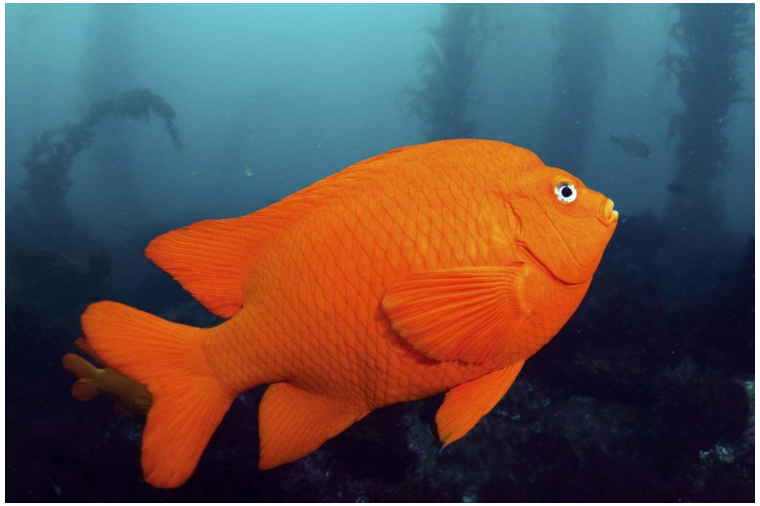

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What is a tensor?

<u>Definition:</u> Let $n \in \mathbb{N}$ and $[n] \coloneqq \{1, \dots, n\}$.

Let $\mathbb{F}$ be a field, and let X, Y, Z be $\mathbb{F}$-vector spaces of dimension at least 2.

A matrix is a 2-way array $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Think of $X \otimes Y$ as the set of $\dim(X) \times \dim(Y)$ matrices

Think of $x \otimes y$ as the array $xy^T = (x_i y_j)_{(i,j)}$

We say that $T \in X \otimes Y$ is product, or rank-one if $T = x \otimes y$ for some $x \in X, y \in Y$

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A ~~matrix~~ tensor is a 3-way array $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix}$

Think of $X \otimes Y \otimes Z$ as the set of $\dim(X) \times \dim(Y) \times \dim(Z)$ tensors

Think of $x \otimes y \otimes z$ as the array $\left(x_i y_j z_k\right)_{(i,j,k)}$

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A ~~matrix~~ tensor is a 3-way array $\begin{bmatrix} 15 & 18 \\ 20 & 24 \end{bmatrix} \begin{bmatrix} 30 & 36 \\ 40 & 48 \end{bmatrix} = \binom{1}{2} \otimes \binom{3}{4} \otimes \binom{5}{6}$

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Let $\mathbb{F}$ be a field, and let X, Y, Z be $\mathbb{F}$-vector spaces of dimension at least 2.

For $T \in X \otimes Y \otimes Z$, an expression $$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z$$

is called a decomposition of T into product tensors

$$\mathrm{rank}(T) := \min\{ n : \text{there exists a decomposition of } T \text{ into } n \text{ product tensors}\}$$

Uniqueness of tensor decompositions

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A tensor decomposition

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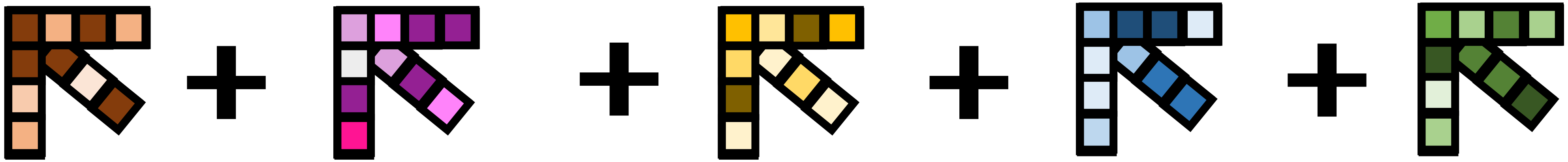

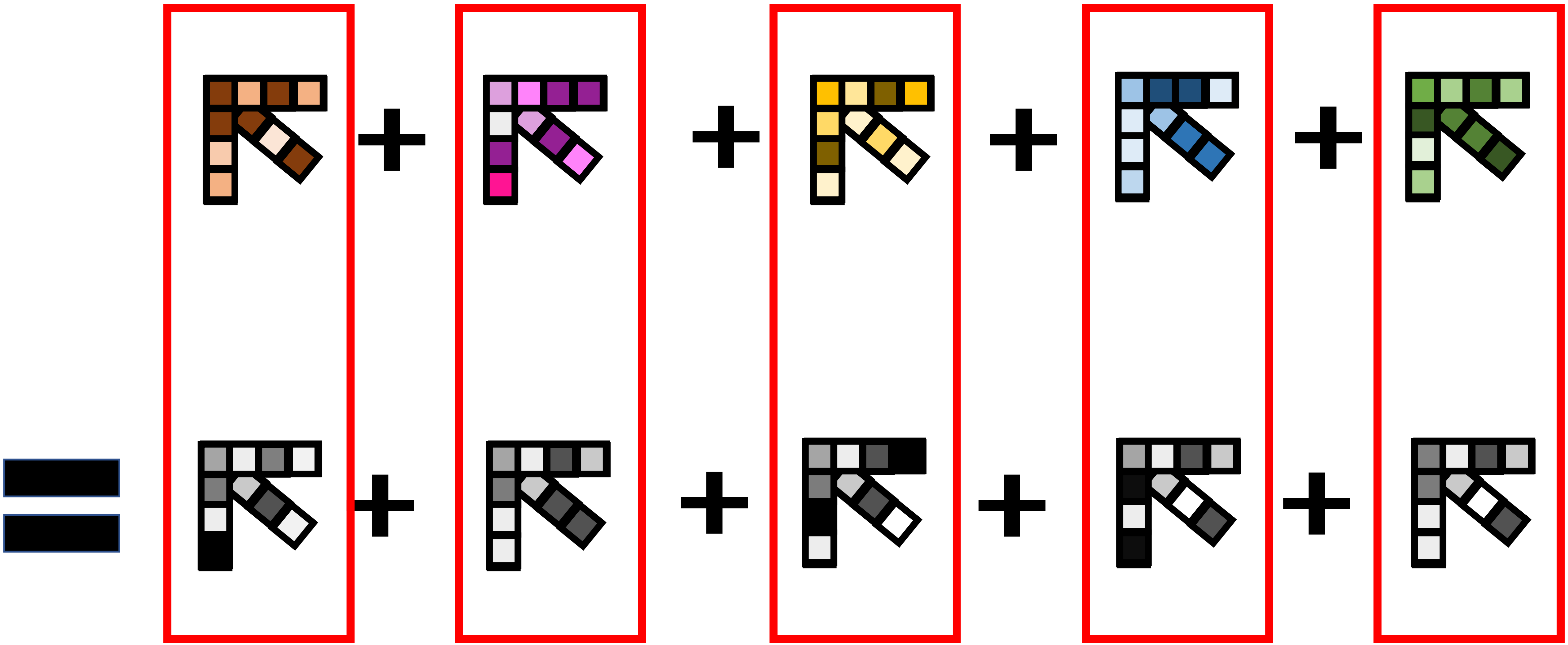

Applications

- Tensors ⟷ Physical data
- Tensor decomposition ⟷ Interpretation of data
- Unique decomposition ⟷ Unique interpretation

Example: Latent parameter learning

L is for *latent*

- Let A, B, C, L be discrete random variables such that A, B, C are conditionally independent, i.e.

$$\Pr(a, b, c|l) = \Pr(a|l)\Pr(b|l)\Pr(c|l) \qquad \text{for all } a, b, c, l.$$

- Goal: Given the probability vector $\Pr(A, B, C)$, determine $\Pr(A, B, C, L)$.
- Method:

$$\Pr(A, B, C) = \sum_l \Pr(l)\Pr(A, B, C|l) = \sum_l \underbrace{\Pr(l)\Pr(A|l) \otimes \Pr(B|l) \otimes \Pr(C|l)}$$

… If $\Pr(A, B, C)$ has a unique decomposition, then we can recover $\Pr(A, B, C, l)$,

- Applications: Learning mixtures of spherical gaussians, phylogenetic tree reconstruction, hidden Markov models, orbit retrieval, blind signal separation, document topic models, …

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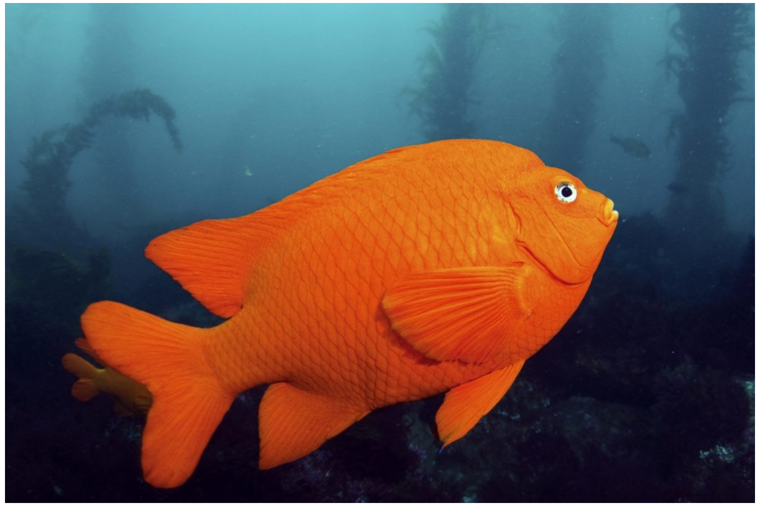

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Kruskal's theorem

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

General flavor of these results:

We are handed a decomposition (1), and we want to know if it is the unique decomposition of T.

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Notation: When $S \subseteq [n]$, $\quad d_x^S := \text{dimspan}\{x_a : a \in S\}$

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$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} + e_4^{\otimes 3} + (e_1 + e_2) \otimes (e_2 + e_3) \otimes (e_1 + e_4)$$

$x_1 \otimes y_1 \otimes z_1$ … x_5 $\otimes$ y_5 $\otimes$ z_5

For $S = \{1,2,5\}$, $\quad d_x^S = 2,\ d_y^S = 3,\ d_z^S = 3$

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$x_1 \otimes y_1 \otimes z_1 \qquad \ldots \qquad x_5 \quad \otimes \quad y_5 \quad \otimes \quad z_5$

$$\{x_1, \ldots, x_5\} = \{e_1, e_2, e_3, e_4, e_1 + e_2\}, \qquad k_x = 2,\ d_x^{[5]} = 4.$$

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<u>Kruskal’s theorem:</u> If $2n \leq k_x + k_y + k_z - 2$, then (1) is the unique decomposition of T.

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Example [Jennrich's Theorem]:

$\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ are linearly independent, and $k_z \geq 2$.

$k_x = k_y = n$

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Weakness: If T is unique under Kruskal's theorem, then so is

$$T' = \sum_{a \in [n]} x_{\sigma(a)} \otimes y_a \otimes z_a \in X \otimes Y \otimes Z$$

for any $\sigma \in S_n$.

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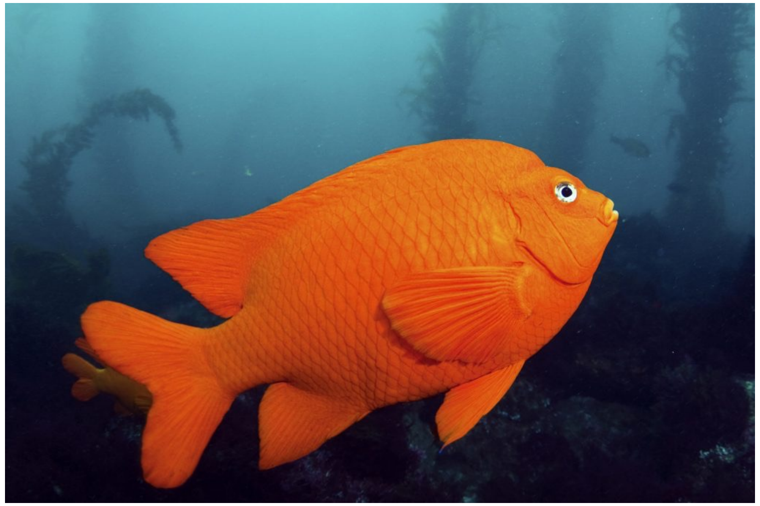

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Our generalization

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

Idea: Replace Kruskal ranks k_x with standard dimspans $d_x^{[n]}$.

<u>Theorem [Gubkin-L-Petrov]</u>: If

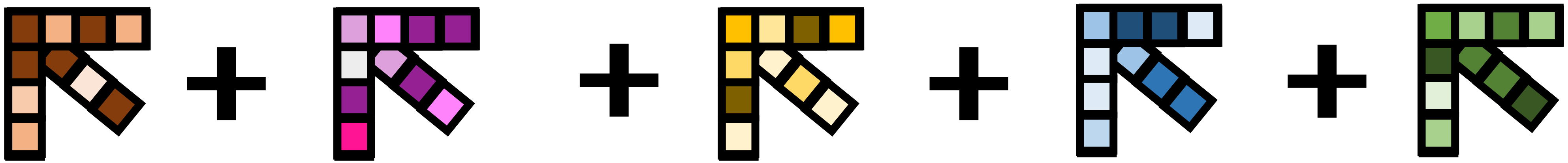
$d_x^{[n]} = \text{dimspan}\{x_1, \dots, x_n\}$

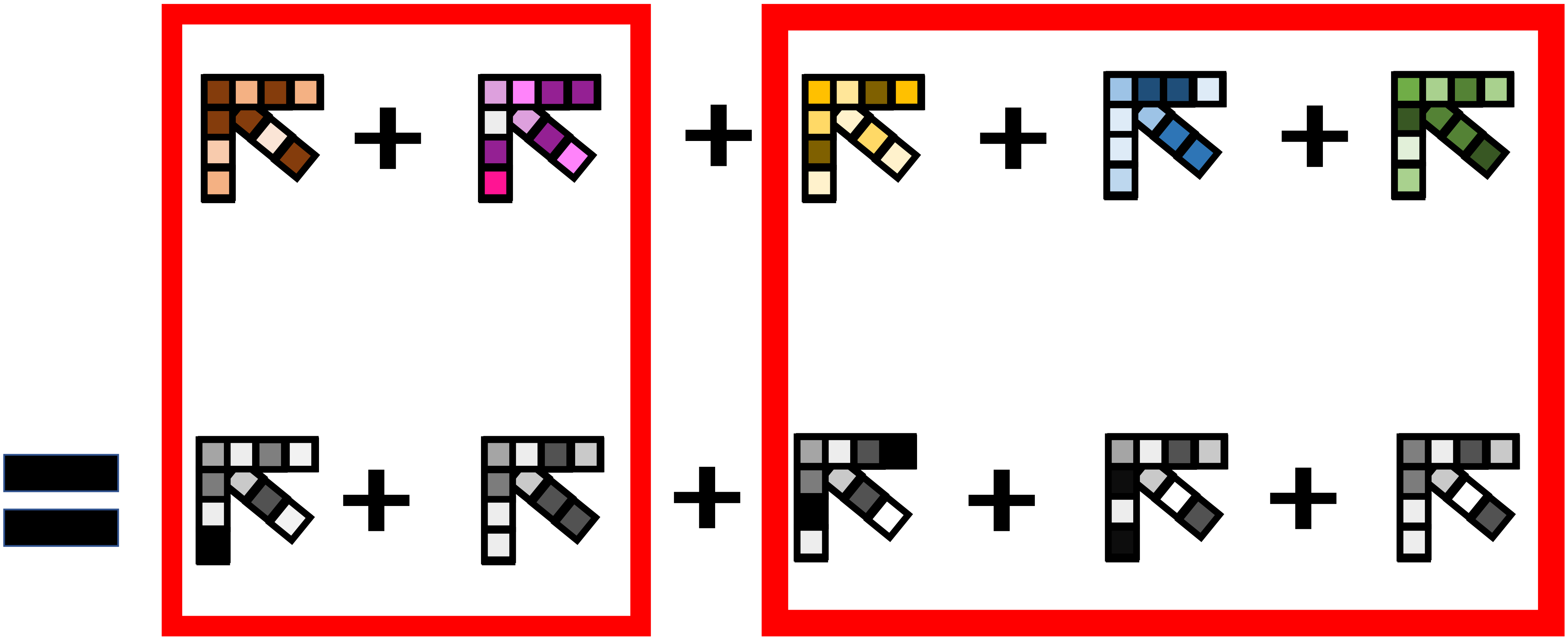
$$2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then for any other decomposition $T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a \in X \otimes Y \otimes Z$

there exist non-trivial subsets $S, R \subseteq [n]$ such that

$$\sum_{a \in S} x_a \otimes y_a \otimes z_a = \sum_{a \in R} x'_a \otimes y'_a \otimes z'_a$$





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Idea: Replace Kruskal ranks k_x with standard dimspans d_x^S.

<u>Theorem [Gubkin-L-Petrov]:</u> If for every subset $S \subseteq [n]$ of size $|S| \geq 2$,

it holds that

$$2|S| \leq d_x^S + d_y^S + d_z^S - 2,$$

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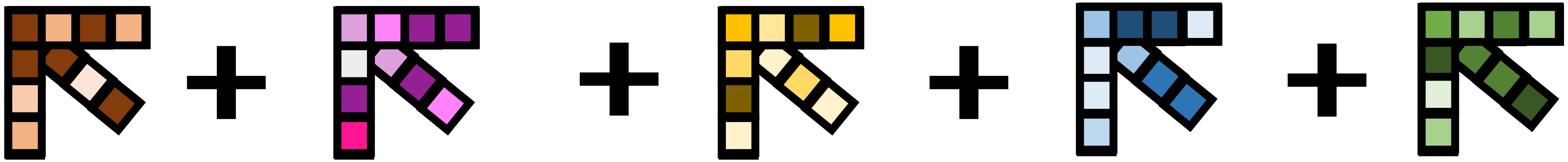
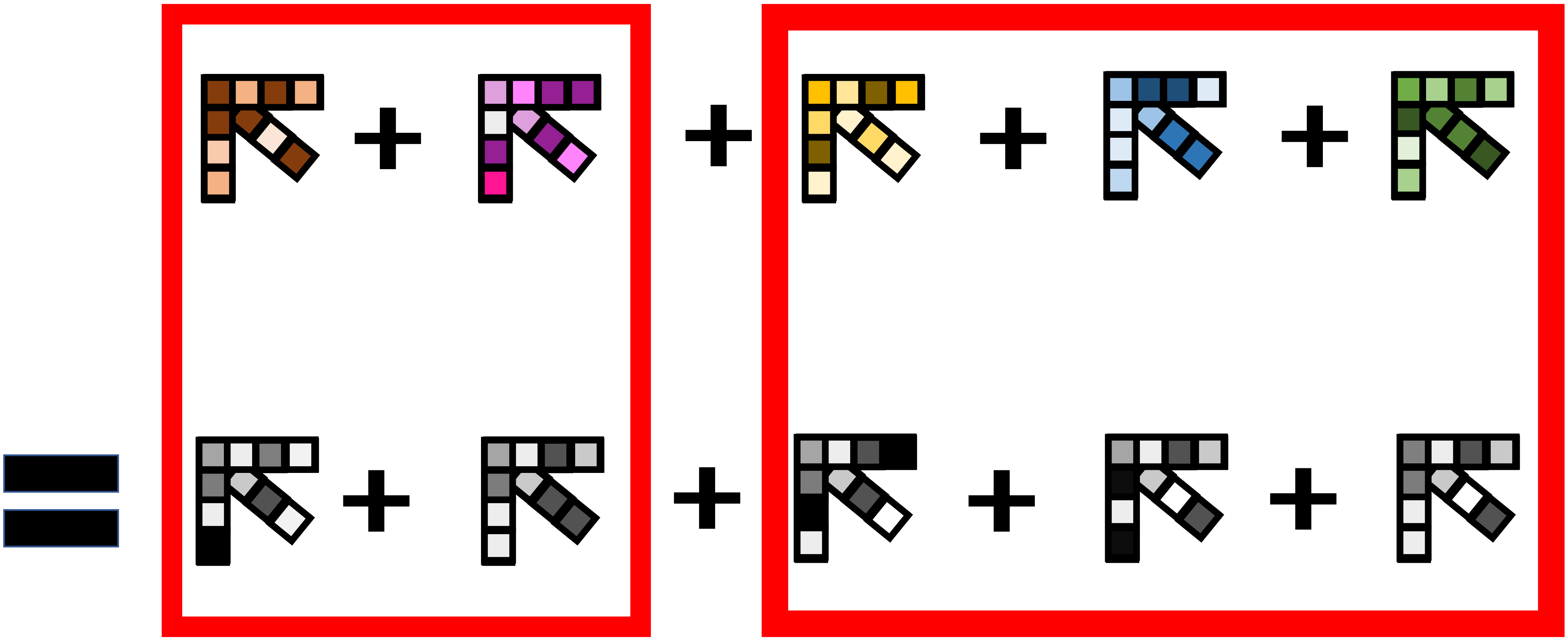
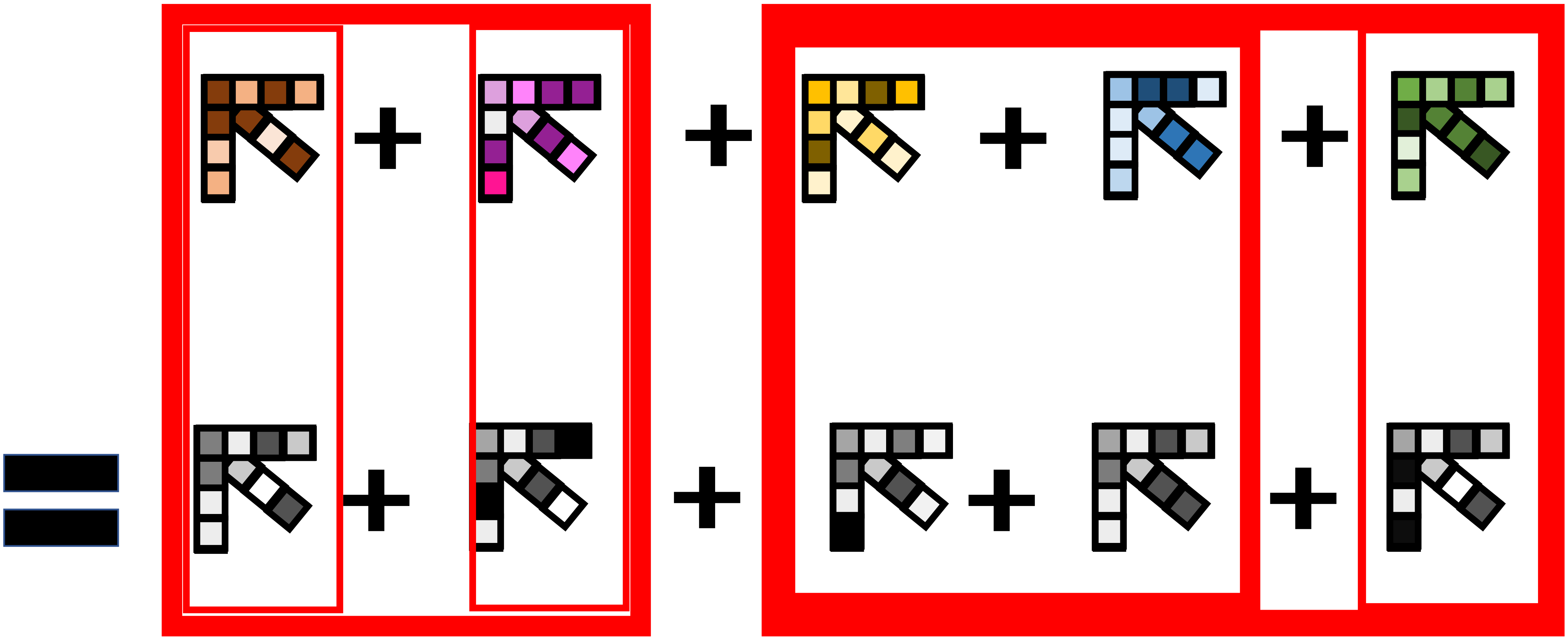
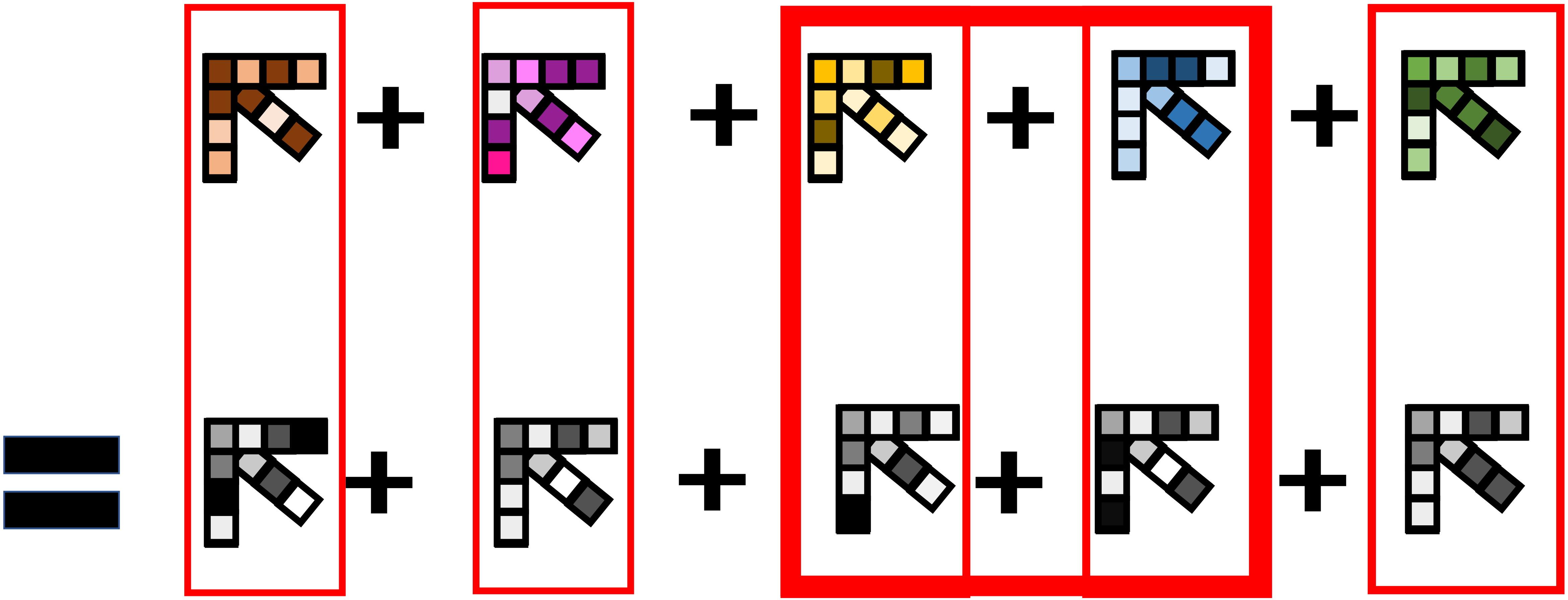
then (1) is the unique decomposition of T.









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Example

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Kruskal's theorem does not certify uniqueness

$$10 = 2n \nleq k_x + k_y + k_z - 2 = 2 + 2 + 2 - 2 = 4$$

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For $S = \{1,2\}$, $4 = 2|S| \leq d_x^S + d_y^S + d_z^S - 2 = 2 + 2 + 2 - 2 = 4$ ✓

For $S = \{1,2,5\}$, $6 = 2|S| \leq d_x^S + d_y^S + d_z^S - 2 = 2 + 3 + 3 - 2 = 6$ ✓

For $S = \{1,2,3,5\}$, $8 = 2|S| \leq d_x^S + d_y^S + d_z^S - 2 = 3 + 3 + 4 - 2 = 8$ ✓

For $S = [5]$, $10 = 2|[n]| \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2 = 4 + 4 + 4 - 2 = 10$ ✓

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Unique by [Gubkin-L-Petrov], but

$$T' = \sum_{a \in [5]} x_{\sigma(a)} \otimes y_a \otimes z_a$$

not unique for $\sigma = (13) \in S_5$

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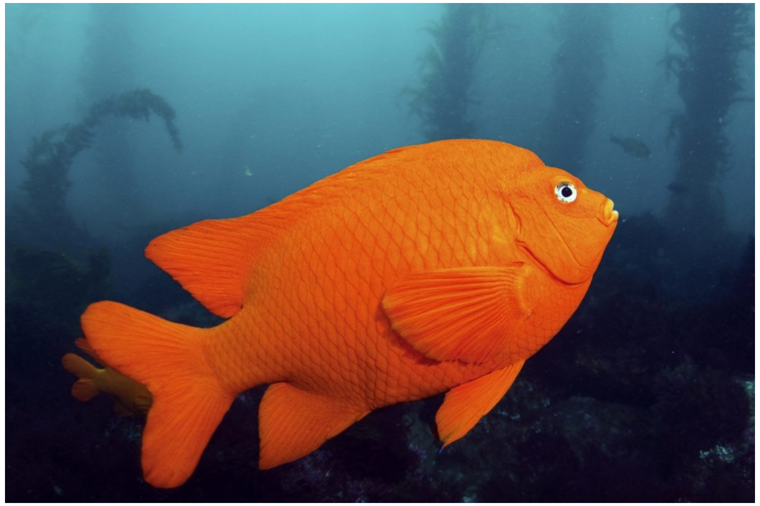

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Splitting

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$$\operatorname{span}(E) = \operatorname{span}(S) \oplus \operatorname{span}(E \setminus S). \tag{2}$$

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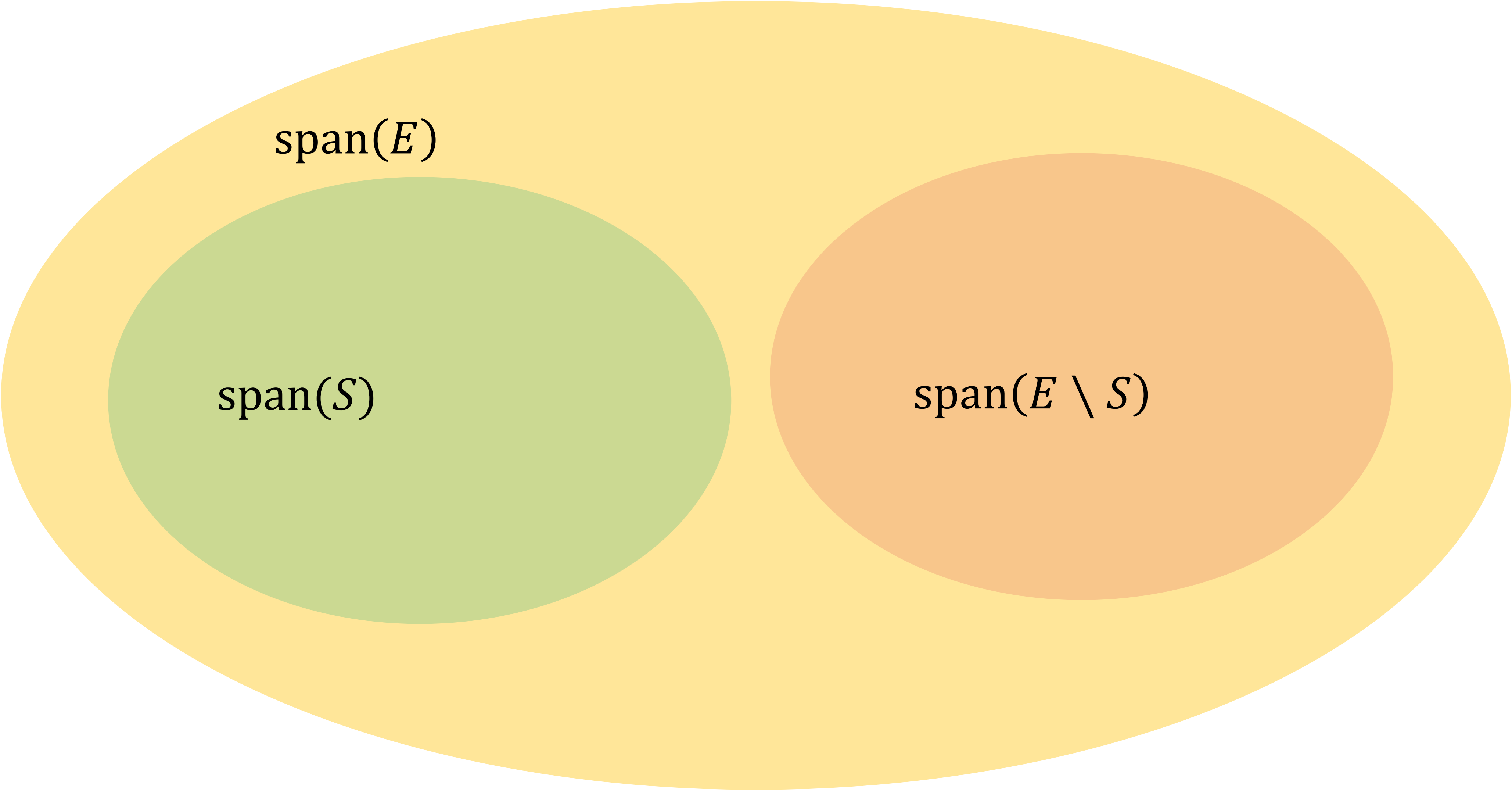
$$\Leftrightarrow \mathrm{span}(S) \cap \mathrm{span}(E \setminus S) = \{0\}$$

E splits if there exists $S \subseteq E$ such that $\text{span}(E) = \text{span}(S) \oplus \text{span}(E \setminus S)$



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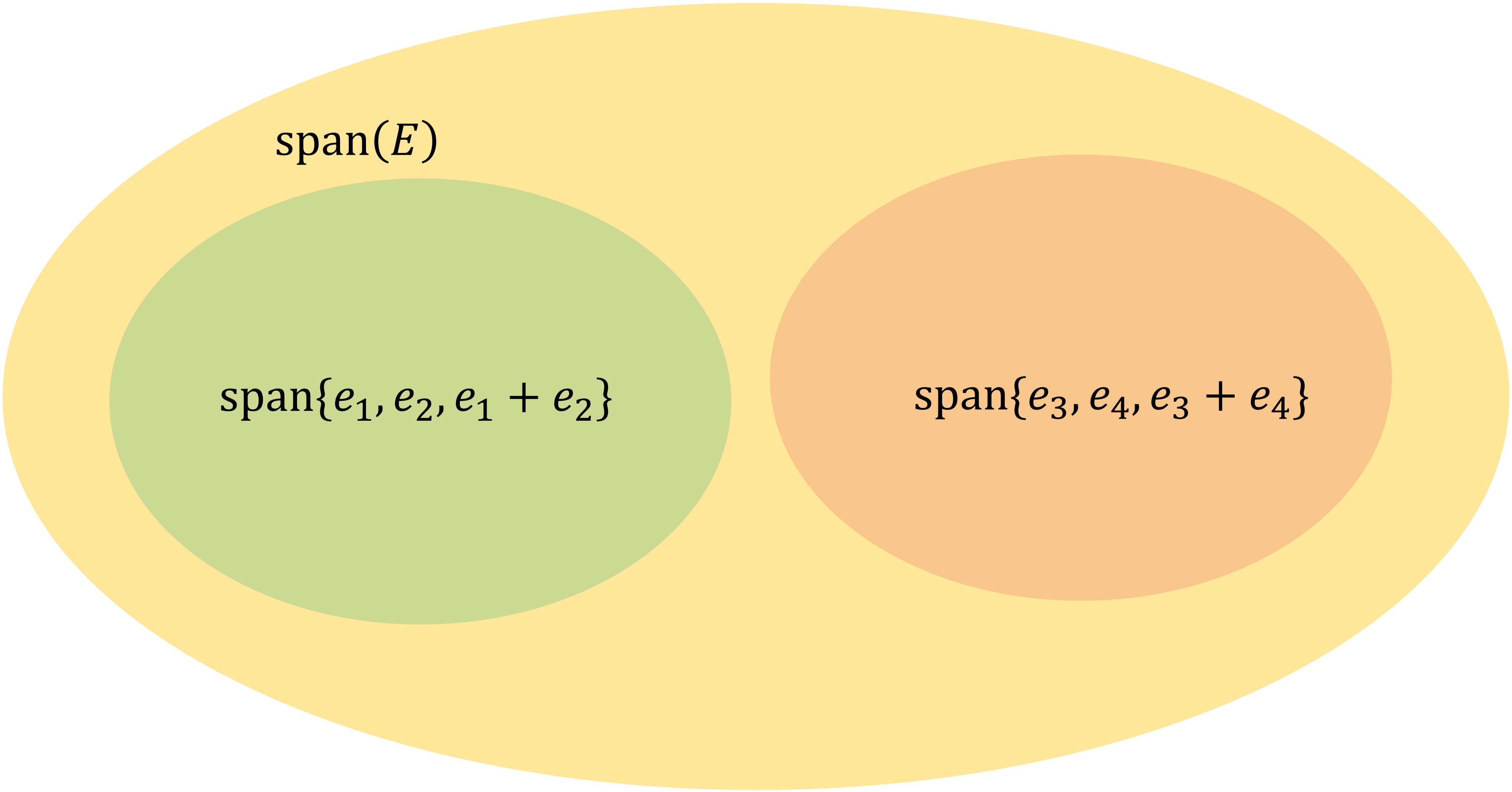

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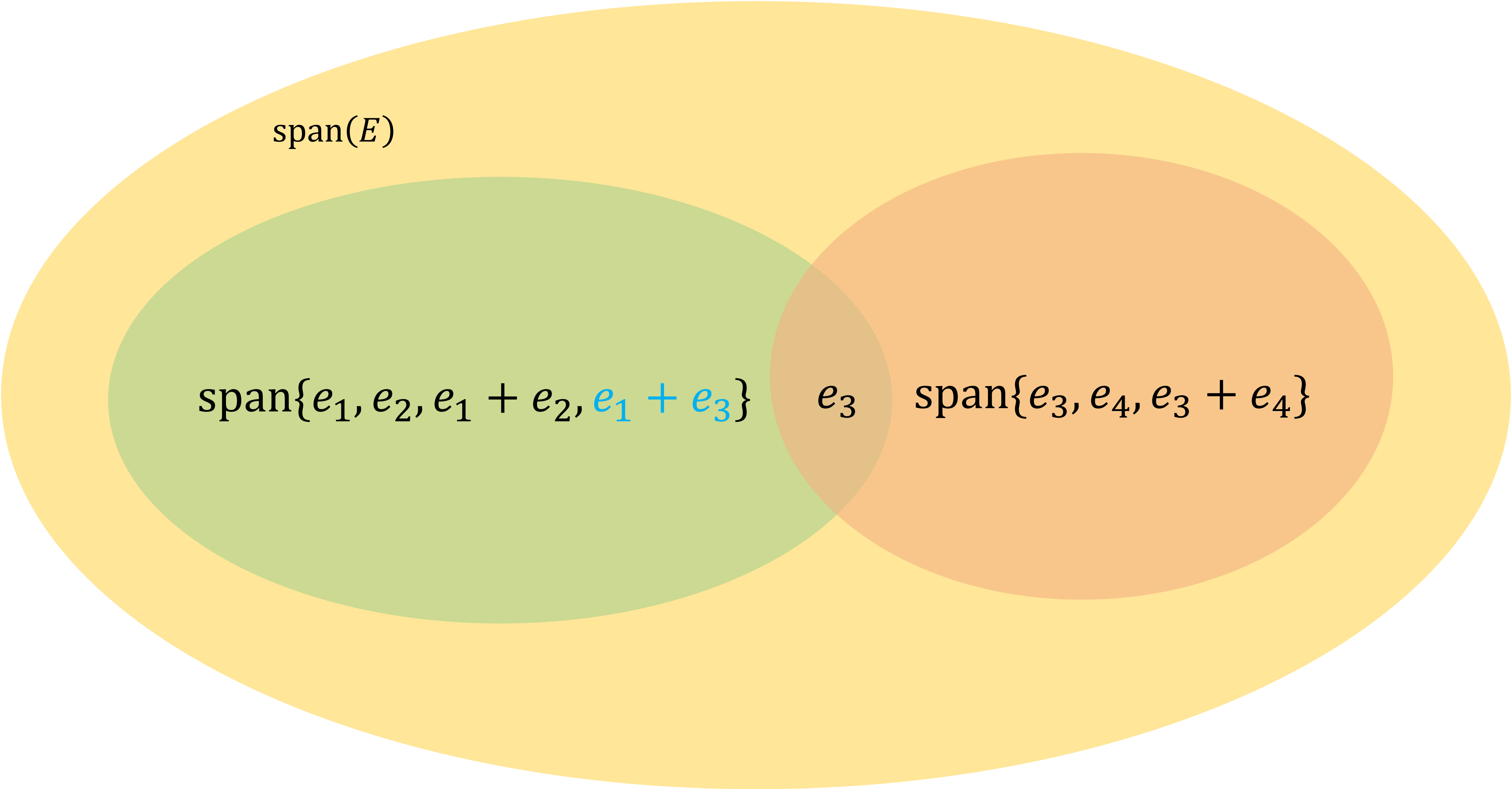

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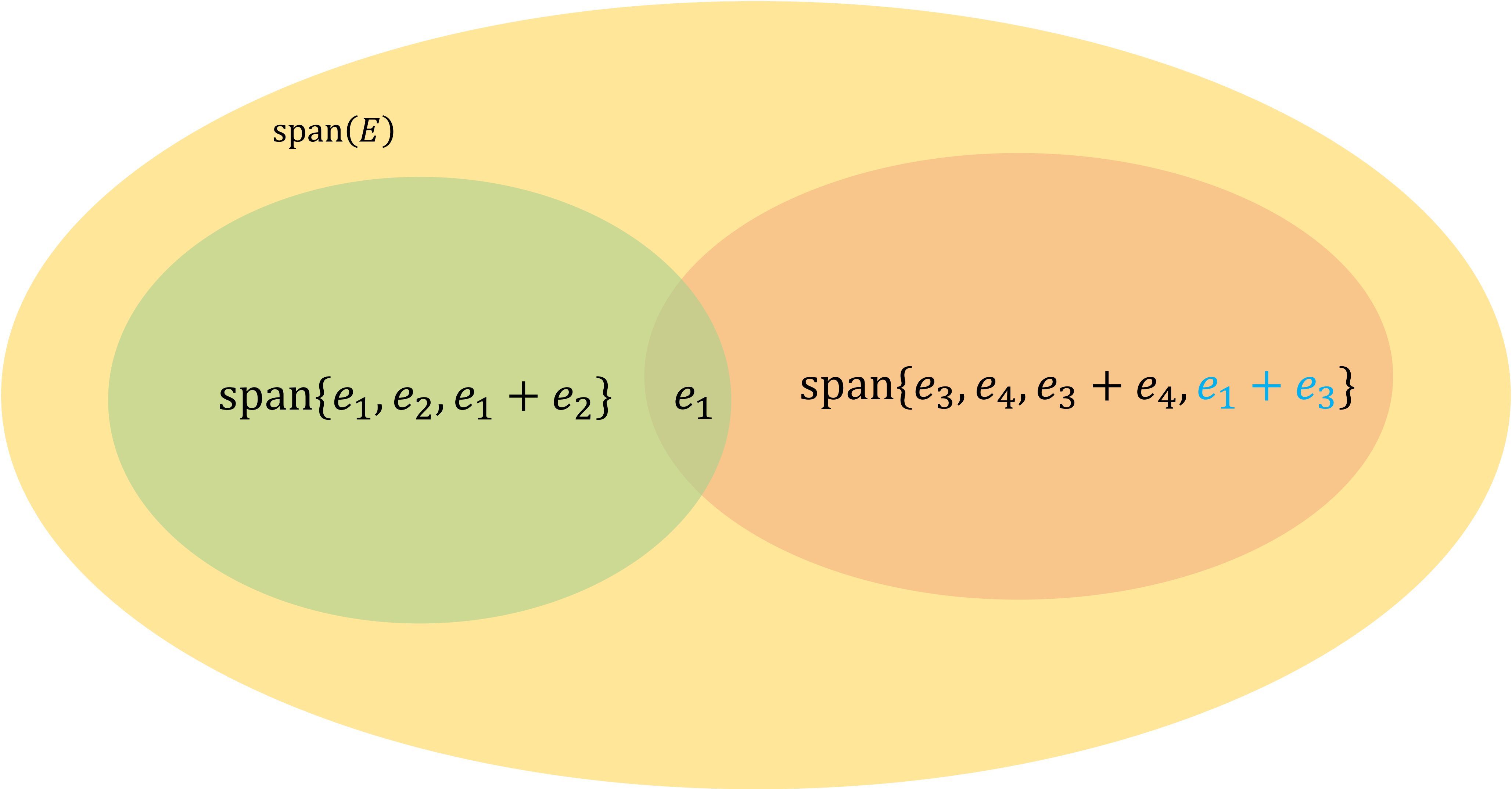

Splitting

Definition: A set of vectors $E = \{v_1, \dots, v_n\}$ splits if there exists a non-trivial subset $S \subseteq E$ such that

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Proof: $\sum(E) = 0 \Rightarrow \sum(S) = -\sum(E \setminus S) \in \operatorname{span}(S) \cap \operatorname{span}(E \setminus S) = \{0\}$ ∎

Splitting theorem

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Splitting theorem [Gubkin-L-Petrov]: Let $E = \{x_a \otimes y_a : a \in [n]\}$.

If

$$\mathrm{dimspan}(E) \leq d_x^{[n]} + d_y^{[n]} - 2$$

then E splits.

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(Our generalization of Kruskal's theorem is a corollary to this)

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How to prove Corollary directly?

Sylvester's rank inequality: $n \le d_x^{[n]} + d_y^{[n]} - \mathrm{rank}(XY^T)$

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Example: $E = \{e_1 \otimes e_1, \quad e_2 \otimes e_2, \quad e_3 \otimes e_2, \quad (e_2 + e_3) \otimes e_2\}$ splits.

$$4 = n \leq d_x^{[4]} + d_y^{[4]} - 1 = 3 + 2 - 1 = 4$$

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Corollary: If

$$n \leq d_x^{[n]} + d_y^{[n]} - 1,$$

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Splitting theorem => Corollary: If E is linearly independent, then it splits. Otherwise,

$\mathrm{dimspan}(E) \leq n - 1 \leq d_x^{[n]} + d_y^{[n]} - 2$, so E splits by splitting theorem.

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If

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Corollary: If

$$n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2,$$

then E splits.

Corollary: If $2n \leq d_x^{[n]} + d_y^{[n]} + d_z^{[n]} - 2$, then for any other set of product tensors $E' = \{x'_a \otimes y'_a \otimes z'_a : a \in [n]\}$, $E \cup E'$ splits.

Corollary => Kruskal generalization

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

Suffices to prove:

<u>Theorem [Gubkin-L-Petrov]:</u> If

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then for any other decomposition $T = \sum_{a \in [n]} x'_a \otimes y'_a \otimes z'_a \in X \otimes Y \otimes Z$

there exist non-trivial subsets $S, R \subseteq [n]$ such that $\sum_{a \in S} x_a \otimes y_a \otimes z_a = \sum_{a \in R} x'_a \otimes y'_a \otimes z'_a$

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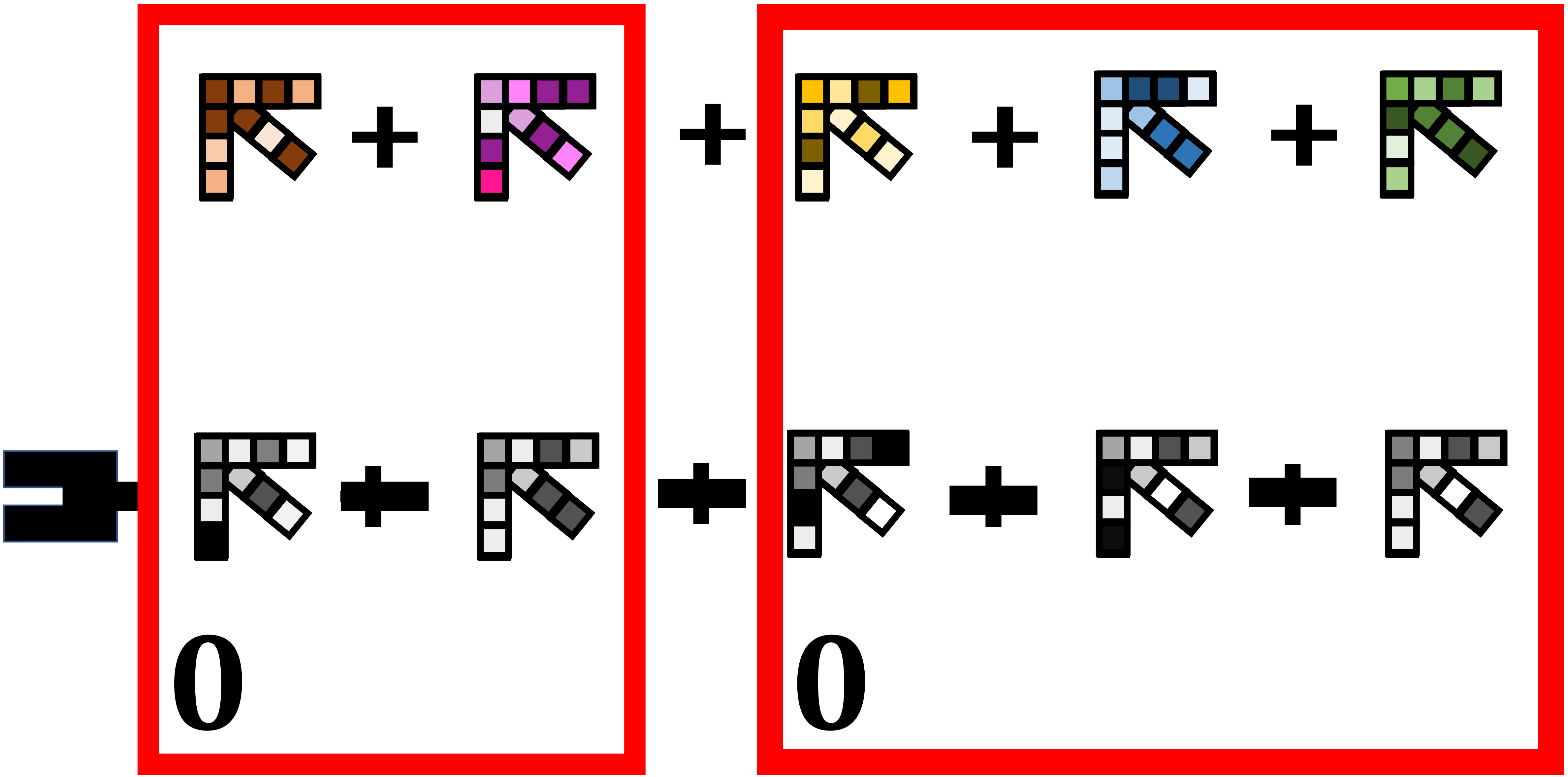
Proof:

By previous corollary, $\{x_a \otimes y_a \otimes z_a, x'_a \otimes y'_a \otimes z'_a \; : a \in [n]\}$ splits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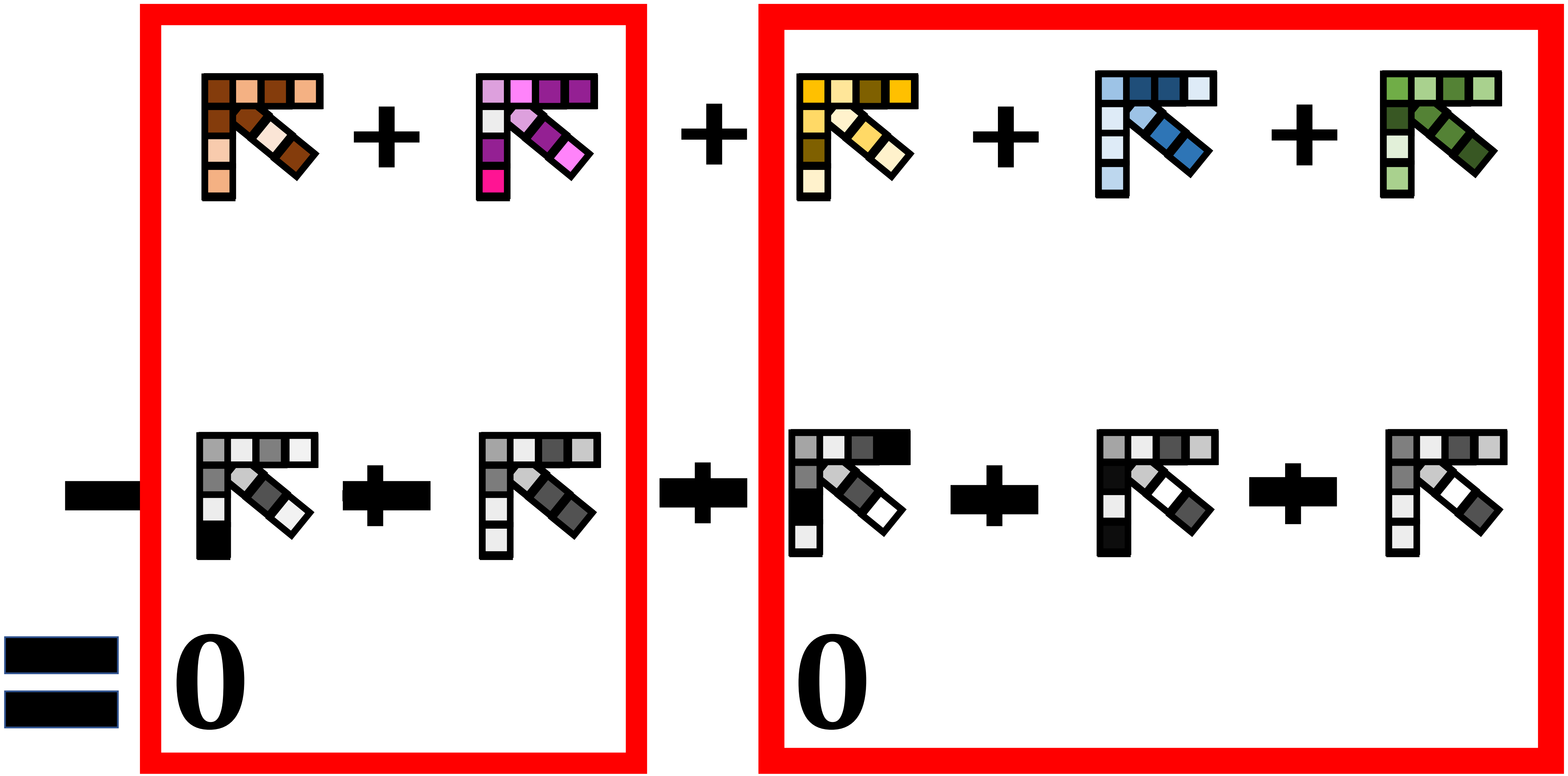

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Outline

- Uniqueness of tensor decompositions
- Kruskal's theorem
- A generalization of Kruskal's theorem
- Splitting theorem
- Another generalization of Kruskal's theorem

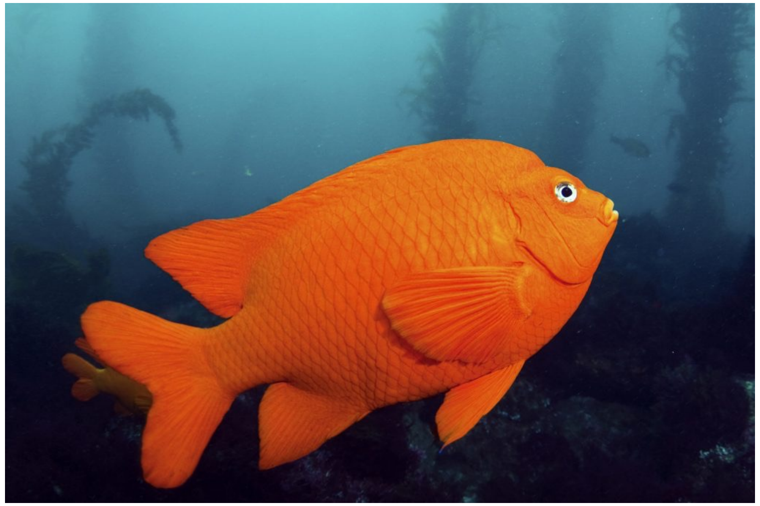

Image: https://fishingbooker.com/blog/california-state-fish-golden-trout-garibaldi/

Other uniqueness results

$$T = \sum_{a \in [n]} x_a \otimes y_a \otimes z_a \in X \otimes Y \otimes Z \qquad (1)$$

1. [Domanov-Sørensen-De Lathauwer, D-De L, S-De L]
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Condition U: $k_x \geq 2$, and for all $\alpha \in \mathbb{F}^n$,

$$\text{rank}\left(\sum_{a \in [n]} \alpha_a \, y_a \otimes z_a\right) \geq \min\{\#\text{nonzero}(\alpha), n - d_x^{[n]} + 2\}.$$

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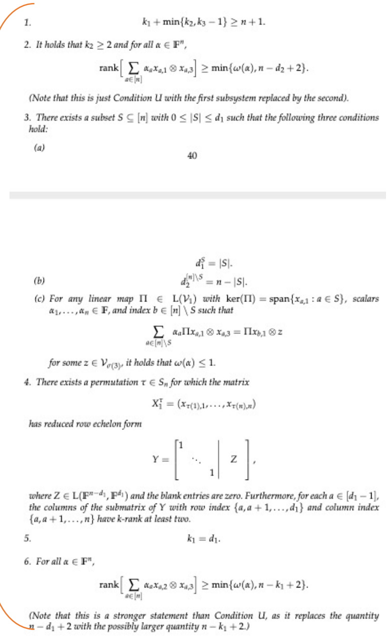
Theorem: If Condition U holds, and any one of the following conditions hold then (1) is the unique decomposition of T.


1. $$k_1 + \min\{k_2, k_3 - 1\} \geq n + 1.$$

2. *It holds that* $k_2 \geq 2$ *and for all* $\alpha \in \mathbb{F}^n$,

$$\operatorname{rank}\Big[\sum_{a\in[n]} \alpha_a x_{a,1} \otimes x_{a,3}\Big] \geq \min\{\omega(\alpha), n - d_2 + 2\}.$$

(Note that this is just Condition U with the first subsystem replaced by the second).

3. *There exists a subset* $S \subseteq [n]$ *with* $0 \leq |S| \leq d_1$ *such that the following three conditions hold:*

   (a) $$d_1^S = |S|.$$

40

   (b) $$d_2^{[n]\setminus S} = n - |S|.$$

   (c) *For any linear map* $\Pi \in \mathrm{L}(\mathcal{V}_1)$ *with* $\ker(\Pi) = \mathrm{span}\{x_{a,1} : a \in S\}$, *scalars* $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, *and index* $b \in [n] \setminus S$ *such that*

$$\sum_{a\in[n]\setminus S} \alpha_a \Pi x_{a,1} \otimes x_{a,3} = \Pi x_{b,1} \otimes z$$

for some $z \in \mathcal{V}_{\sigma(3)}$, *it holds that* $\omega(\alpha) \leq 1$.

4. *There exists a permutation* $\tau \in S_n$ *for which the matrix*

$$X_1^\tau = (x_{\tau(1),1}, \ldots, x_{\tau(n),n})$$

has reduced row echelon form

$$Y = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & Z \\ & & 1 & \end{array}\right],$$

where $Z \in \mathrm{L}(\mathbb{F}^{n-d_1}, \mathbb{F}^{d_1})$ *and the blank entries are zero. Furthermore, for each* $a \in [d_1 - 1]$, *the columns of the submatrix of* Y *with row index* $\{a, a+1, \ldots, d_1\}$ *and column index* $\{a, a+1, \ldots, n\}$ *have k-rank at least two.*

5. $$k_1 = d_1.$$

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(Note that this is a stronger statement than Condition U, as it replaces the quantity $n - d_1 + 2$ *with the possibly larger quantity* $n - k_1 + 2$.*)*

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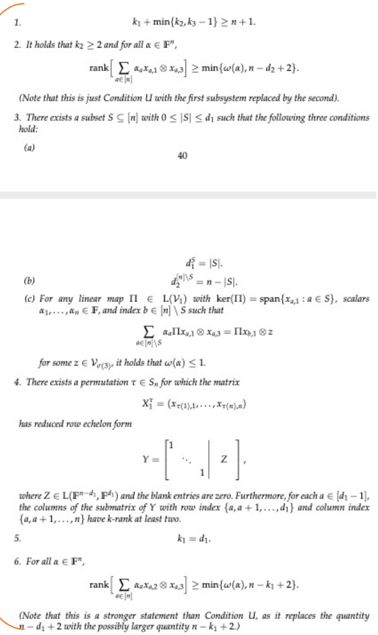

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(Note that this is a stronger statement than Condition U, as it replaces the quantity $n - d_1 + 2$ *with the possibly larger quantity* $n - k_1 + 2$.*)*

Proof sketch: By Kruskal’s permutation lemma, Condition U implies that the x_a’s are unique. Use extra assumptions to prove full uniqueness. ∎

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This also generalizes Kruskal: If $2n \leq k_x + k_y + k_z - 2$, then Condition U holds.

… but it has similar drawbacks [D-De L]: Condition U implies $\min\{k_y, k_z\} \geq n - d_x^{[n]} + 2$.

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Theorem [Gubkin-L-Petrov]: If Condition U holds, then (1) is the unique decomposition of T.

Summary [Gubkin-L-Petrov]:

Condition U	⟹	Unique
⇑̸ ⇓̸		
$2\lvert S\rvert \leq d_x^S + d_y^S + d_z^S - 2$ for all $S \subseteq [n]$ s.t $\lvert S\rvert \geq 2$	⟹	Unique

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Related: Uniqueness of symmetric decompositions, generic uniqueness, uniqueness for low rank tensors, finite decompositions,…

Conclusion

- Uniqueness of tensor decompositions
- Kruskal’s theorem
- A generalization of Kruskal’s theorem
- Splitting theorem
- Another generalization of Kruskal’s theorem
- More matroid theory for product tensors?

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